trigonometry of the ellipse; and a true insight into the relation between the circular and hyperbolic functions can be attained only by moving the ellipse into the place of honor, and studying the functions that naturally present themselves in connection with its equation.

The results of an investigation into this matter proved to be highly interesting. First of all, the primary or ancestral forms for both circular and hyperbolic functions were brought to light; and secondly, the trigonometry of all the central conics, real or imaginary, presented itself in a systematic manner, and without reference to De Moivre's theorem. In fact, the ancestral type of De Moivre's formula was discovered at once.

A previous attempt at a generalization in this field was made by Prof. Irving Stringham in the *American Journal of Mathematics*, vol. 14, 1892, and in a little book entitled *Uniplanar Algebra* (Univ. of California Press). He lays down two entirely arbitrary definitions:

$$\sin_k w = \frac{k}{2}(e^{w/k} - e^{-w/k}), \qquad \cos_k w = \tfrac{1}{2}(e^{w/k} + e^{-w/k}).$$

These, with four others derived from them ($\tan_k$, $\cot_k$, $\sec_k$, $\csc_k$), he calls "modocyclic" functions; for $k = 1$, we get the hyperbolic functions, for $k = i$ the circular. Stringham's $\sin_k$ and $\cos_k$ are mutually unsymmetric, whereas the circle and hyperbola are symmetric in two ways; his formulas are therefore not the ultimate ancestral types; they are realizable in the curve $x^2 + y^2/k^2 = 1$. though he did not advert to this fact. Later on, Becker and Van Orstrand (*Smithsonian Tables of Hyperbolic Functions*, 1909) used this $k$-ellipse in developing their theory; but the presence of only one parameter makes it impossible to derive formulas in this way for the general central conic. Our formulas will have to have two parameters corresponding to the two parameters of the locus

1. *The "Protocyclic" functions.*

Consider the hyperbola

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1;$$

the area $APM$ is given by

$$\frac{b}{a}\int_x^a \sqrt{x^2 - a^2}\,dx = \frac{b}{2a}\left(x\sqrt{x^2 - a^2} - a^2 \log \frac{x + \sqrt{x^2 - a^2}}{a}\right)$$

$$= OPM - OAP;$$

hence if $OAP$ is called $H$, we have

$$\frac{2H}{ab} = \log\left(\frac{x}{a} + \frac{y}{b}\right) \qquad \text{or} \qquad \frac{x}{a} + \frac{y}{b} = e^{2H/ab}.$$

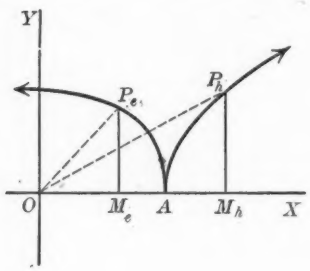
The curve-equation furnishes as the factorial mate to this

$$\frac{x}{a} - \frac{y}{b} = e^{-(2H/ab)};$$

so that

$$x_h = \frac{a}{2}(e^{2H/ab} + e^{-(2H/ab)}), \qquad y_h = \frac{b}{2}(e^{2H/ab} - e^{-(2H/ab)}).$$

Consider in the next place, the ellipse




$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1;$$

its area $APM$ is

$$\frac{b}{a}\int_a^x \sqrt{a^2 - x^2}\,dx = \frac{b}{ai}\int_a^x \sqrt{x^2 - a^2}\,dx.$$

Hence if $OAP$ is called $E$, we get as above

$$\frac{2E}{ab}i = \log\left(\frac{x}{a} + \frac{iy}{b}\right),$$

and

$$x_e = \frac{a}{2}(e^{(2E/ab)i} + e^{-(2E/ab)i}), \qquad y_e = \frac{b}{2i}(e^{(2E/ab)i} - e^{-(2E/ab)i}).$$

Comparing these $E$ and $H$ formulas, we observe that if we put

$$2H \text{ (or } 2E) = w, \qquad \frac{1}{a} = k_1, \qquad \text{and} \qquad \frac{1}{b}\left(\text{or } \frac{i}{b}\right) = k_2,$$

the expressions for $x$ become identical; likewise those for $y$; and the equation of both curves becomes $k_1^2x^2 - k_2^2y^2 = 1$. We therefore lay down the following proposition:

*Every central conic $q_1^2x^2 - q_2^2y^2 = 1$ has associated with it two functions, which define $x$ and $y$ for any point $P$ on the curve in terms of the area of the vertex-sector $OAP = \Omega/2$. These functions are called the cosine, and the sine, respectively; and are defined by the equations*

$$(1) \qquad x = \cos\Omega = \frac{e^{q_1q_2\Omega} + e^{-q_1q_2\Omega}}{2q_1}; \qquad y = \sin\Omega = \frac{e^{q_1q_2\Omega} - e^{-q_1q_2\Omega}}{2q_2}.^1$$

[1] In the case of the circle and equilateral hyperbola this proposition has been a familiar one of our text books—for example, *An elementary treatise on the differential calculus* by J. M. Rice and W. W. Johnson, New York, 1880, pp. 139–140; according to Montucla (*Histoire des mathématiques*, Vol. 3, 1802, p. 287) the conception dates back to the time of Lambert. The extension of this idea as above to the general conic by means of an imaginary integration for the ellipse, and the consequent development of a train of two-parameter formulas including those for the circle and hyperbola as sub-cases carries this process to its ultimate completion in the most direct manner. A generalization from circle and equilateral hyperbola to any ellipse and any hyperbola was considered by C. A. Laisant, *Essai sur les fonctions hyperboliques*, Paris, 1874, p. 37 f. (also in *Mémoires de la société des sciences de Bordeaux*, Vol. 10). In particular Laisant indicates the relations between $v$ and $w$ (see page 286 of this paper), in which "the figure is unified in a remarkable manner." Professor Frumveller's presentation of his rediscovery of this and other relations, and of his deductions therefrom, is well worthy of perusal, even by those to whom Laisant's scarce volume is accessible. See also S. Günther, *Die Lehre von den gewöhnlichen und verallgemeinerten Hyperbelfunktionen*, 1881, p. 371 f.—EDITOR.

These, with their quotients and inverses, constitute the six functions that we shall call "protocyclic"; the number $\Omega$ will be taken to be essentially real, at least for the present, while $q_1$, $q_2$, are numbers of the general type $u + iv$. If the reference-curve be $\varphi(q_1, q_2) = 0$, the four simplest classes of these functions are those associated with the curves $\varphi(k_1, k_2)$; $\varphi(k_1, ik_2)$; $\varphi(ik_1, k_2)$; $\varphi(ik_1, ik_2)$; the two first mentioned degenerating into the ordinary hyperbolic and circular functions respectively, when $k_1 = k_2 = 1$ and both are real.

Solving equations (1) for the exponentials, and raising both sides to the $n$th power, we obtain the most general form of DeMoivre's theorem, valid for all $n$'s;

$$(2) \qquad e^{\pm q_1 q_2 n\Omega} = q_1 \cos n\Omega \pm q_2 \sin n\Omega = (q_1 \cos \Omega \pm q_2 \sin \Omega)^n.$$

When $n$ is negative, we must note that $\cos \Omega = \cos(-\Omega)$, since the cosine is an even function, while the sine is odd. In like manner

$$(3) \qquad \prod_{\lambda=1}^{m} (q_1 \cos \Omega_\lambda \pm q_2 \sin \Omega_\lambda) = q_1 \cos\left(\sum_1^n \Omega_\lambda\right) \pm q_2 \sin\left(\sum_1^n \Omega_\lambda\right),$$

as is evident on changing both sides into exponentials.

The equation of the curve, together with the definitions (1), gives

$$(4) \qquad \begin{aligned} &q_1^2 \cos^2 \Omega - q_2^2 \sin^2 \Omega = 1, \text{ with its corollaries,} \\ &q_1^2 \qquad - q_2^2 \tan^2 \Omega = \sec^2 \Omega, \qquad q_1^2 \cot^2 \Omega - q_2^2 \qquad = \csc^2 \Omega. \end{aligned}$$

No end of formulas may now be evolved from these data! Let $w$ and $w_1$ be two special values of $\Omega$; form the four products $\sin w \cos w_1$, $\cos w \sin w_1$, $\sin w \sin w_1$, $\cos w \cos w_1$, using exponentials as in equations (1); then

$$(5) \qquad \begin{aligned} \sin(w \pm w_1) &= q_1(\sin w \cos w_1 \pm \cos w \sin w_1), \\ \cos(w \pm w_1) &= (1/q_1)(q_1^2 \cos w \cos w_1 \pm q_2^2 \sin w \sin w_1). \end{aligned}$$

From these we get tangent and secant formulas, formulas for $w/2$, $2w$, $3w$, etc., in the usual way. On adding these two sine or cosine equations, and writing $\xi = w + w_1$, $\eta = w - w_1$, product-formulas result:

$$(6) \qquad \begin{aligned} \sin \xi \pm \sin \eta &= 2q_1 \sin\left(\frac{\xi \pm \eta}{2}\right) \cos\left(\frac{\xi \mp \eta}{2}\right), \\ \cos \xi + \cos \eta &= 2q_1 \cos\left(\frac{\xi + \eta}{2}\right) \cos\left(\frac{\xi - \eta}{2}\right), \\ \cos \xi - \cos \eta &= 2\frac{q_2^2}{q_1} \sin\left(\frac{\xi + \eta}{2}\right) \sin\left(\frac{\xi - \eta}{2}\right). \end{aligned}$$

To find the general formulas for $\sin^m w$ and $\cos^m w$ in multiples of $w$, we write $\xi^r = q_1 \cos rw + q_2 \sin rw$, and $1/\xi^r = q_1 \cos rw - q_2 \sin rw$, so that

$$\cos rw = \frac{1}{2q_1}\left(\xi^r + \frac{1}{\xi^r}\right), \qquad \sin rw = \frac{1}{2q_2}\left(\xi^r - \frac{1}{\xi^r}\right).$$

Thus

$$\cos^{2n} w = \left(\frac{1}{2q_1}\right)^{2n}\left(\xi + \frac{1}{\xi}\right)^{2n}$$

$$= \left(\frac{1}{2q_1}\right)^{2n}\left[\left(\xi^{2n} + \frac{1}{\xi^{2n}}\right) + \binom{2n}{1}\left(\xi^{2n-2} + \frac{1}{\xi^{2n-2}}\right) + \cdots\right].$$

A similar expansion is found for $\sin^m w$, the results being

$$\cos^{2n} w = \left(\frac{1}{2q_1}\right)^{2n-1}\left[\cos 2nw + \binom{2n}{1}\cos(2n-2)w + \binom{2n}{2}\cos(2n-4)w + \cdots + \frac{1}{2q_1}\binom{2n}{n}.\right]$$

$$\cos^{2n+1} w = \left(\frac{1}{2q_1}\right)^{2n}\left[\cos(2n+1)w + \binom{2n+1}{1}\cos(2n-1)w + \cdots + \binom{2n+1}{n}\cos w\right]$$

(7)

$$\sin^{2n} w = 2q_1\left(\frac{1}{2q_2}\right)^{2n}\left[\cos 2nw - \binom{2n}{1}\cos(2n-2)w + \binom{2n}{2}\cos(2n-4)w - \cdots + (-1)^n\frac{1}{2q_1}\binom{2n}{n}\right]$$

$$\sin^{2n+1} w = \left(\frac{1}{2q_2}\right)^{2n}\left[\sin(2n+1)w - \binom{2n+1}{1}\sin(2n-1)w + \cdots + (-1)^n\binom{2n+1}{n}\sin w\right].$$

These will suffice to give an idea of what the new formulas look like; $q_1 = 1$, $q_2 = i$, reduces them all to the case of ordinary trigonometry.

Since our fundamental $\Omega$-functions are exponential, they are *periodic:* their period is $2\pi i/q_1q_2$, so that

$$f(q_1q_2\Omega) = f[q_1q_2(\Omega \pm 2n\pi i/q_1q_2)]; \qquad f(q_1q_2\Omega/i) = f[q_1q_2(\Omega/i \pm 2n\pi/q_1q_2)].$$

To establish this, we observe that when $q_1q_2\Omega$ becomes $i\Omega$ by putting $q_1 = 1, q_2 = i$, the $x$ and $y$ of formulas (1) become the ordinary trigonometric functions $\cos_c$, $\sin_c$, associated with $x^2 + y^2 = 1$; thus

$$y_c = \sin_c \Omega = (1/2i)(e^{i\Omega} - e^{-i\Omega}) = \sin_c(\Omega \pm 2n\pi) = (1/2i)(e^{i(\Omega\pm 2n\pi)} - e^{-i(\Omega\pm 2n\pi)});$$

in general therefore, we get $\sin_c$, $\cos_c$, by placing $q_1q_2\Omega = i(\Omega \pm 2n\pi)$, *i. e.*, by placing

$$q_1q_2\left(\Omega \pm \frac{2n\pi i}{q_1q_2}\right) = i\Omega.$$

It is now an easy matter to follow the variation of our general functions sin, cos, tan, etc., through quarter-periods;

$$
\begin{aligned}
&\text{if } q_1q_2\Omega = \frac{\pi}{2}i, \quad \left(\text{or } \Omega = \frac{\pi i}{2q_1q_2}\right), \quad \sin\Omega = \frac{i}{q_2}\sin_c\frac{\pi}{2} = \frac{i}{q_2};\\
&\text{if } q_1q_2\Omega = \frac{2\pi}{2}i, \quad \left(\text{or } \Omega = \frac{2\pi i}{2q_1q_2}\right), \quad \sin\Omega = \frac{i}{q_2}\sin_c\frac{2\pi}{2} = 0;\\
&\text{if } q_1q_2\Omega = \frac{3\pi}{2}i, \quad \left(\text{or } \Omega = \frac{3\pi i}{2q_1q_2}\right), \quad \sin\Omega = \frac{i}{q_2}\sin_c\frac{3\pi}{2} = -\frac{i}{q_2};\\
&\text{if } q_1q_2\Omega = \frac{4\pi}{2}i, \quad \left(\text{or } \Omega = \frac{4\pi i}{2q_1q_2}\right), \quad \sin\Omega = \frac{i}{q_2}\sin_c\frac{4\pi}{2} = 0.
\end{aligned}
\tag{8}
$$

The corresponding cycle for $\cos\Omega$ is $(0, -1/q_1, 0, 1/q_1)$; from these the tangent and secant follow at once. The tangent has a period $\pi i/q_1q_2$.

The combined use of formulas (5) and (8) now gives us the generalized forms of the trigonometric expressions for $\sin_c(90 \pm A)$, $\cos_c(180 \pm A)$, etc., *i.e.*, for *complements* and *supplements* of $\Omega$; the following are specimens:

$$
\begin{aligned}
&\sin\left(\Omega \pm \frac{\pi i}{2q_1q_2}\right) = \pm\frac{iq_1}{q_2}\cos\Omega; \qquad \sin\left(\Omega \pm \frac{2\pi i}{2q_1q_2}\right) = -\sin\Omega,\\
&\cos\left(\Omega \pm \frac{\pi i}{2q_1q_2}\right) = \pm\frac{iq_2}{q_1}\sin\Omega; \qquad \cos\left(\Omega \pm \frac{2\pi i}{2q_1q_2}\right) = -\cos\Omega.
\end{aligned}
\tag{9}
$$

We might mention in passing that if $w$ is a value of $\Omega$,

$$\frac{d}{dw}\sin w = {q_1}^2\cos w,$$

and

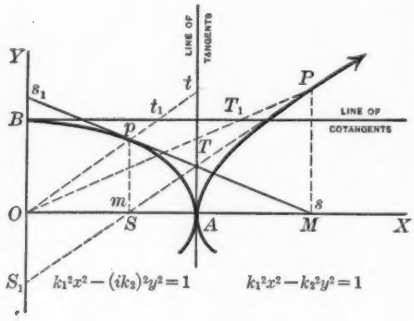
$$\frac{d}{dw}\cos w = {q_2}^2\sin w,$$

so that our sin and cos functions satisfy the differential equation $D_w{}^2z - {q_1}^2{q_2}^2z = 0$. As a final rounding out of our theory, we give the series-expansions of these two functions:

$$\sin w = \frac{1}{q_2}\sum_0^\infty \frac{(q_1q_2w)^{2n+1}}{\underline{|2n+1}}; \qquad \cos w = \frac{1}{q_1}\sum_0^\infty \frac{(q_1q_2w)^{2n}}{\underline{|2n}}. \tag{10}$$

## 2. *Geometry of the protocyclic functions.*

If $k_1$, $k_2$, are real numbers, the reference-curve ${q_1}^2x^2 - {q_2}^2y^2 = 1$ has four fundamental types, namely $(q_1, q_2) = (k_1, k_2)$, $(k_1, ik_2)$, $(ik_1, k_2)$, or $(ik_1, ik_2)$; the first and second form a pair (hyperbola and ellipse), likewise the third and fourth. Each of these types has a geometrical representation for the six functions; and the geometry of each pair has cross-relationships of a very interesting kind.




We shall use the subscripts $e$, $h$, $c$, for the functions connected with the ellipse, hyperbola, and circle.

(*a*) *Types* $(k_1, k_2)$ *and* $(k_1, ik_2)$. In the adjacent hyperbola let $w = 2(OAP)$; $AT, PS$, are tangents to $H$; $OA = 1/k_1$; $OB = 1/k_2$; $PS \equiv k_1^2xx_h - k_2^2yy_h - 1 = 0$; $OP \equiv y - (\sin_h w/\cos_h w)x = 0$. The various intercepts of these lines give at once the following table of line-values for the protocyclic functions of a point $P$ on $H$:

$$PM = \sin_h w, \qquad OS_1 = \frac{-1}{k_2^2}\csc_h w,$$

$$OM = \cos_h w, \qquad OS = \frac{1}{k_1^2}\sec_h w,$$

$$AT = \frac{1}{k_1}\tan_h w, \qquad BT_1 = \frac{1}{k_2}\cot_h w.$$

In the same way precisely, the ellipse $E$ has the following line-values at the point $p$;

$$pm = \sin_e v, \qquad Os_1 = \frac{1}{k_2^2}\csc_e v,$$

$$Om = \cos_e v, \qquad Os = \frac{1}{k_1^2}\sec_e v,$$

$$At = \frac{1}{k_1}\tan_e v, \qquad Bt_1 = \frac{1}{k_2}\cot_e v.$$

When $p$ and $P$ are chosen at random, there is no relation between $v$ and $w$; but if ordinate $pm$ be erected at the point $S$, the entire figure is unified in a remarkable manner.[1] We have first of all, $OS = \cos_e v = (1/k_1^2)\sec_h w$; $ps$, the tangent at $p$, has for its $x$-intercept, $1/(k_1^2x_e) = (1/k_1^2)\sec_e v = \cos_h w$, so that it passes through $M$. Using formulas (4) of § 1 on the two values of $OS$, we have

$$(1/k_1^2)(1 - k_2^2\sin_e^2 v) = (1/k_1^4)(k_1^2 - k_2^2\tan_h^2 w)$$

whence

$$\sin_e v = \left(\frac{1}{k_1}\right)\tan_h w;$$

applying

$$\cos_e v = \left(\frac{1}{k_1^2}\right)\sec_h w$$

to this, we get $\tan_e v = k_1 \sin_h w$; which shows that the points $t$ and $P$, $T$ and $p$,

[1] Cf. footnote on page 282.

are respectively at the same level above $OX$. The relation thus existing between $v$ and $w$ is expressed by the functional symbol $v_e = \text{am } w_h$, or $v_e = \text{gd } w_h$ (read, "$v$ is the *amplitude* or *gudermannian*[1] of $w$").

The operation "$gd$" is carried out as follows, by means of formula 2, § 1;

$$e^{k_1 k_2 w} = k_1 \cos_h w + k_2 \sin_h w = k_1^{-1} \sec_e v + (k_2/k_1) \tan_e v = V,$$

whence

$$w = \frac{1}{k_1 k_2} \log V.$$

If there were in existence a table of the various functions of $v_e$, we could consequently compute $w_h$ and its functions from that table; ordinary hyperbolic functions for $k_1 = k_2 = 1$ are thus easily found from the corresponding tables of circular trigonometry.[2]

(*b*) *Types* $(ik_1, k_2)$ *and* $(ik_1, ik_2)$. The fundamental loci now become

$$\begin{cases} H_{-1}; & (ik_1)^2x^2 - (ik_2)^2y^2 = 1, \quad \text{or} \quad -k_1^2x^2 + k_2^2y^2 = 1. \\ E_{-1}; & (ik_1)^2x^2 - k_2^2y_2 = 1, \quad \text{or} \quad -k_1^2x^2 - k_2^2y^2 = 1. \end{cases}$$

The ellipse $E_{-1}$ is the imaginary form of our previous $E$; hence $H_{-1}$ must be regarded as the imaginary form of $H$. All the formulas of § 1 were predicated on the supposition that $\Omega$ was *real*; but we may generalize our point of view as regards $x$ and $y$ in the following manner: "*the theory as well as the geometry of the protocyclic functions remain valid, even when x or y is a pure imaginary number.*"

Graphing under such circumstances has been fully worked out by the writer in a former paper[3] to which the reader must be referred for details.

The equation $H_{-1}$, or $-k_1^2x^2 + k_2^2y^2 = 1$, when $x = u + iv$, and $y$ is real, becomes $k_2^2y^2 = 1 + k^2_1(u^2 - v^2)$, with the condition $uv = 0$; if $v = 0$,

$$k_2^2y^2 - k_1^2u^2 = 1$$

in the plane $\overline{uoy}$; if $u = 0$, $k_2^2y^2 + k_1^2v^2 = 1$ in plane $\overline{voy}$. The other equation $E_{-1}$, or $k_1^2x^2 + k_2^2y^2 = -1$, gives $1 = -k_1^2u^2 - k_2^2y^2$ in $\overline{uoy}$, and $1 = k_1^2v^2 - k_2^2y^2$ in $\overline{voy}$; in the complex plane $\overline{voy}$, we therefore find the geometrical locus of case (*a*) reproduced exactly!

When the $k$'s become complex numbers, we can still write

$$(H) \qquad y^2 = \frac{1}{k_2^2} - \left(\frac{k_1^2}{k_2^2}\right)x^2 = (r_1 + is_1) - (r_2 + is_2)(u + iv)^2$$

$$= (r_1 - r_2u^2 + r_2v^2 + 2s_2uv) + i(s_1 - s_2u^2 + s_2v^2 - 2r_2uv),$$

[1] The term "Gudermannian" was introduced by Cayley in a paper of 1862 (see *Collected Mathematical Papers*, Volume 5, p. 86); if $u = \log \tan [(\pi/4) + (\phi/2)]$, then $\phi = 1/i \log \tan [(\pi/4) + \frac{1}{2}ui] =$ "gd $u$ (Gudermannian of $u$)." The name was given in honor of Gudermann who called the function the "longitude of $u$" (Crelle's *Journal*, Vol. 6, 1830, p. 165).—EDITOR.

[2] The *Smithsonian Tables* referred to above contain an extensive table of the gudermannian (and another of the antigudermannian); a brief one may be found in B. O. Peirce, *A Short Table of Integrals*, revised edition.—EDITOR.

[3] "The graph of $f(x)$ for complex numbers," AMERICAN MATH. MONTHLY, Vol. 24, November, 1917.

with a similar equation for $(E)$, but there is no simple geometrical method of attacking this wilderness of ordinates. The $y$'s stand at every point in the $x$-plane, but they have various slants towards the fourth direction of space; all we notice is, that those of the same slant or tilt have their feet on an hyperbola around $O$ as center. Methods of function-theory must here be used.

*Conclusion.*—One primary advantage of treating hyperbolic functions from this general standpoint lies, in the writer's judgment, in the relegation of $i$ to its proper subordinate position; protocyclic functions are real, and represent geometric entities from the very start; the generalization that occurs in their definition is seen to be geometrically appropriate. The theoretic interest is mainly in the fact that circle-trigonometry and hyperbola-trigonometry can now be comprehended in the wider topic of the trigonometry of the central conics.

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## A PROOF OF A THEOREM OF COMPOUND PROBABILITIES.

By UGO BROGGI,[1] Universidad De La Plata.

1. The definition of a geometric probability is well known. We call the probability, $\phi(a, b)$, that a real number, $x$, belongs to the interval $(a, b)$ a real function, $\phi$, of $a$ and $b$ which satisfies the conditions

$$0 \leq \phi(a, b) \leq 1,$$

and, if $a \leq c \leq b$,

$$\phi(a, b) = \phi(a, c) + \phi(c, b).$$

It would be an easy matter, but beside our purpose, to generalize the definition, and apply it not only to intervals but also to more general one- or more-dimensional sets of points.

Since the definition of our new probability is based upon the theorem of total probabilities, it is evident that it must agree with that theorem.

We cannot without proof say the same of the theorem of compound probabilities, which asserts that if $p$ is the probability that $x$ belongs to $(a, b)$, and $\pi$ the probability that $y$ belongs to $(\alpha, \beta)$ and there is no relation between $x$ and $y$, the probability $P$ that the two-dimensional number $(x, y)$ belongs to the region

$$a \leq x \leq b, \qquad \alpha \leq y \leq \beta,$$

is expressed by $p\pi$.

It is a surprising fact that nowhere do we find the demonstration of this theorem, one which is evidently fundamental and without which the theory of errors and the kinetic theory of gases would come to nothing.

We propose to give here the missing demonstration.

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[1] Professori Broggi is a native of Italy, received his doctorate at Göttingen (1907) and has long been professor of mathematics at the University of La Plata in Buenos Aires. His dissertation was entitled *Die Axiome der Wahrscheinlichkeitsrechnung* and his work *Matematica Attuariale* (Milano, 1906) was issued in French form in 1907, and in German in 1911.—Editor.

2. Suppose $a$ and $\alpha$ to be constant, and put

$$b = a + h, \quad \beta = \alpha + k; \quad p = p(a, h), \quad \pi = \pi(\alpha, k); \quad P = P(a, \alpha; h, k).$$

It is evident that

$$p \geq P, \qquad \pi \geq P,$$

and, when $h_2 > h_1 > 0$, $k_2 > k_1 > 0$,

$$p(a, h_2) - p(a, h_1) \geq P(a, \alpha; h_2, k) - P(a, \alpha; h_1, k) \geq 0$$

$$\pi(\alpha, k_2) - \pi(\alpha, k_1) \geq P(a, \alpha; h, k_2) - P(a, \alpha; h, k_1) \geq 0.$$

$P$ cannot be an increasing function of $h$ or of $k$ unless $p$ and $\pi$ are increasing functions of their arguments. If $p$ and $\pi$ are increasing functions of $h$ and of $k$, there is only one value of $h$ or of $k$ corresponding to a given value of $p$ or of $\pi$, respectively. Hence in

$$P = P(a, \alpha; h, k)$$

we can express $h$ and $k$ through $p$ and $\pi$ and have

$$P = F(p, \pi).$$

If

$$\pi_1 = \pi(\alpha, k_1), \qquad \pi_2 = \pi - \pi_1 \qquad (k_1 \leq k)$$

we have also, from the theorem of total probabilities

$$P = F(p, \pi_1) + F(p, \pi_2)$$

and therefore[1]

$$P = p\pi.$$

3. Let us now suppose that the inequality

$$p(a, h_1) < p(a, h_2) \qquad (h_1 < h_2)$$

holds true only for $h \leq h_1$, and that two values $h_1$ and $h_2$ exist, such that

$$p(a, h_1) = p(a, h_2).$$

Then for every $h$ belonging to the interval $(h_1, h_2)$ we must have

$$p(a, h) = p(a, h_1).$$

$$0 = p(a, h) - p(a, h_1) \geq P(a, \alpha; h, k) - P(a, \alpha; h_1, k) \geq 0$$

$$P(a, \alpha; h, k) = P(a, \alpha; h_1, k) = p(a, h)\pi(\alpha, k).$$

The theorem must still hold true when we have identically

$$p(a, h) = 0$$

[1] Broggi, "Il teorema della probabilità composta," in *Rendiconti del Circolo Matematico di Palermo*, Vol. 28 (1909), 245–247.

for $h \leq h_1$. In this case

$$p \geq P \geq 0, \qquad P = p\pi = 0.$$

4. We can summarize the results here obtained by saying that the theorem of compound probabilities holds true when $p$ and $\pi$ are increasing functions of their arguments, or increase in certain intervals and are constant elsewhere. But this condition is always realized, as we cannot have

$$p(a, h_1) = p(a, h_2), \qquad \pi(\alpha, k_1) = \pi(\alpha, k_2)$$

unless the functions $p$ and $\pi$ are constant in the intervals $(h_1, h_2)$ and $(k_1, k_2)$ respectively.

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## BITS OF HISTORY ABOUT TWO COMMON MATHEMATICAL TERMS.

By G. A. MILLER, University of Illinois.

In 1841 A. L. Cauchy defined the term *indicator* (indicateur) corresponding to a given modulus $n$ as the exponent to which a positive integer $m$ relatively prime to $n$ belongs mod $n$, and gave tables for the determination of the maximum indicator $I$ corresponding to a given modulus $n$, where $n$ may be replaced successively by a series of positive integers. For instance,

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n =$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| $I =$ | 1 | 2 | 2 | 4 | 2 | 6 | 2 | 6 | 4 | 10 | 2 | 12 |

About four years later E. Prouhet published a note in volume 5 of the *Nouvelles Annales de Mathématiques*, page 75, in which he defined the term indicator of $n$ as the number $\phi(n)$ of the positive integers less than $n$ and prime to $n$. He added incorrectly that this term had not been employed previously in mathematics. It seems somewhat singular that this later definition came into common use, especially since Cauchy was a more noted mathematician than Prouhet and this definition was due to ignorance on the part of its author.

In the language of group theory Cauchy's definition of indicator is equivalent to that of the order of the totitives of $n$, while that of Prouhet is equivalent to that of the order of the group formed by these totitives mod $n$. The $I$ of $n$ is therefore always a divisor of $\varphi(n)$ and a necessary and sufficient condition that $I$ of $n$ is equal to $\phi(n)$ is that $n$ has a primitive root, or that the group of totitives of $n$ is cyclic. In general, the value of $I$ is equivalent to the order of the largest cyclic subgroup contained in this group.

As an instance where a later definition of a common term had an entirely different fate we may refer to the definition of *simple group* found on page 65, volume 20, *Proceedings of the London Mathematical Society*. According to this definition a group is called *simple* when it is both cyclic and its order is a power of a prime number, while the common definition of simple group as a group which

does not involve any invariant subgroup besides the identity had been used by C. Jordan about twenty years earlier.

No one else seems to have adopted this later definition of simple group. Its implicit use in L. E. Dickson's *History of the Theory of Numbers*, volume 1, page 131, is doubtless due to an oversight. Curiously H. W. L. Tanner continued the use of his unfortunate definition in an article published about seven years later in volume 27 of the same journal, page 331.

The mathematical literature contains numerous definitions due to a lack of knowledge on the part of their authors. The different fates of the two cited above may serve to illustrate how difficult it is to deal with some of these definitions. The fact that they are potential sources of error, especially when they appear in periodicals which are frequently used, makes it undesirable to ignore even those which are not adopted by others.

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## QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

### REPLIES.

**36.** A number of Discussions have been published in this department relating to cubic and biquadratic equations (cf. Vol. XXXV, p. 29, pp. 268–269 and 343–347; Vol. XXIV, pp. 136–137 and 436–439; Vol. XXIII, pp. 314–315). Below are given a number of questions sent in by Professor Harris Hancock of the University of Cincinnati which relate to the cubic and biquadratic and might, perhaps, more properly be proposed as problems were it not for the advantage to be gained, if possible, by treating them all, or at least several of them, in one discussion.

1. For what values of $n$ can $\cos 2\pi/n$ be expressed in the form $(a + \sqrt{b})/c$ where $a$, $b$ and $c$ are integers?

2. Write the biquadratic in the form $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$. Show that its reducing cubic may be expressed by means of a determinant of the third order which when expanded is

$$4y^3 - g_2 y - g_3 = 0,$$

where $g_2 = ae - 4bd + 3c^2$, and

$$g_3 = \begin{vmatrix} a, b, c \\ b, c, d \\ c, d, e \end{vmatrix} = ace - ad^2 - eb^2 - c^3 + 2bcd.$$

3. For the same biquadratic show that

$$a^3(x_0 + x_1 - x_2 - x_3)(x_0 + x_2 - x_1 - x_3)(x_0 + x_3 - x_1 - x_2) = 32(3abc - a^2d - 2b^3)$$

without making any use of symmetric functions, where $x_0, \cdots, x_3$ are the roots of the biquadratic equation.

4. If $x_0, x_1, x_2$ are the roots of a cubic and $D$ its discriminant, show that $x_1$ is a rational function of $x_0$ and $\sqrt{D}$. Derive a much simpler relation than that given in Serret, *Cours d'Algèbre Supérieure*, 5th ed., No. 511.

5. If $x_0, x_1, x_2, x_3$ are the roots of a biquadratic, $D$ its discriminant, $e_1, e_2, e_3$ the roots of the reducing cubic, show that $x_1$ is a rational function of $x_0, e_1, e_2, e_3$ and consequently also of $x_0, e_1$, and $\sqrt{D}$.

6. If the biquadratic

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0 \qquad (1)$$

has a double root, show that the reducing cubic (cf., for example, Burnside and Panton's *Theory of Equations*)

$$t^3 + 3Ht^2 + \left(3H^2 - \frac{a^2 I}{4}\right)t - \frac{G^2}{4} = 0 \tag{2}$$

has a root in common with the cubic

$$8t^3 + 12Ht^2 + G^2 = 0, \qquad (G \neq 0); \tag{3}$$

and conversely, if (2) and (3) have a common root, then (1) has a double root.

## Reply by Harris Hancock, University of Cincinnati.

The following reply deals with parts 2–6 of Question 36.

2. With Descartes write the biquadratic $f(x) = ax^4 + 4bx^3 + 6cx^2 + 4dx + e$ in the form $(px^2 + 2qx + r)(p'x^2 + 2q'x + r')$, where

$$pp' = a, \qquad pq' + qp' = 2b, \qquad pr' + rp' + 4qq' = 6c, \qquad qr' + rq' = 2d, \qquad rr' = e.$$

The third of these expressions becomes $pr' + rp' = 2c - 4s$, if we put $qq' = c + s$. Note that the determinant

$$\begin{vmatrix} 2pp', & pq' + qp', & pr' + rp' \\ qp' + q'p, & 2qq', & qr' + rq' \\ rp' + r'p, & rq' + r'q, & 2rr' \end{vmatrix}$$

is the product

$$\begin{vmatrix} p, & p', & 0 \\ q, & q', & 0 \\ r, & r', & 0 \end{vmatrix} \begin{vmatrix} p', & p, & 0 \\ q', & q, & 0 \\ r', & r, & 0 \end{vmatrix} = 0.$$

Substituting the above values in the first determinant, we have

$$\begin{vmatrix} a, & b, & c - 2s \\ b, & c + s, & d \\ c - 2s, & d, & e \end{vmatrix} = 0;$$

or,

$$4s^3 - g_2 s - g_3 = 0,$$

where

$$g_2 = ae - 4bd + 3c^2,$$

and

$$g_3 = \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}.$$

This form of the reducing cubic is well known. It is very important in the Theory of Elliptic Integrals and should be given in every text-book on the subject.

3. Let $x_0, x_1, x_2, x_3$ be the roots of the above quartic. It follows that

$$\frac{f(x) - f(x_0)}{x - x_0} = a(x - x_1)(x - x_2)(x - x_3).$$

Writing

$$x = -x_0 - \frac{2b}{a} = \tfrac{1}{2}(x_1 + x_2 + x_3 - x_0),$$

this expression becomes

$$\frac{f\left(-x_0 - \frac{2b}{a}\right) - f(x_0)}{2x_0 + \frac{2b}{a}} = \frac{a}{8}P,$$

where

$$P = (x_0 + x_1 - x_2 - x_3)(x_0 + x_2 - x_1 - x_3)(x_0 + x_3 - x_1 - x_2).$$

Note that $P$ remains unchanged when we interchange $x_0$ with $x_1$, or with $x_2$, or with $x_3$. It follows

that the equation of the third degree

$$\frac{f\left(-x-\frac{2b}{a}\right)-f(x)}{ax+b}=\tfrac{1}{4}P,$$

holds for four values of $x$, and consequently for every value of $x$. Hence, writing $x=0$, it is seen that

$$\frac{8}{a^3}(3abc-a^2d-2b^3)=\tfrac{1}{4}P.$$

In a similar manner it may be proved that

$$f'(x_0)+f'(x_1)+f'(x_2)+f'(x_3)=P.$$

4. Write the cubic in the form

$$K(x)=x^3-p_1x^2+p_2x+p_3\equiv(x-x_0)(x-x_1)(x-x_2)=0,$$
$$K'(x_0)=(x_0-x_1)(x_0-x_2),$$
$$\sqrt{D}=(x_1-x_0)(x_2-x_0)(x_2-x_1).$$

It follows that

$$x_2-x_1=\frac{\sqrt{D}}{K'(x_0)};$$

and since $x_2+x_1=p_1-x_0$, we have

$$2x_2=p_1-x_0+\frac{\sqrt{D}}{K'(x_0)},\qquad 2x_1=p_1-x_0-\frac{\sqrt{D}}{K'(x_0)}.$$

Similarly[1] it is seen that

$$2x_2=p_1-x_1-\frac{\sqrt{D}}{K'(x_1)},\qquad 2x_0=p_1-x_1+\frac{\sqrt{D}}{K'(x_1)};$$

$$2x_0=p_1-x_2-\frac{\sqrt{D}}{K'(x_2)},\qquad 2x_1=p_1-x_2+\frac{\sqrt{D}}{K'(x_2)}.$$

5. In Burnside and Panton, *Theory of Equations*, fourth edition, p. 124, it is shown that the roots of the above biquadratic may be written in the form

$$ax_0+b=\sqrt{e_1}+\sqrt{e_2}+\sqrt{e_3}=z_0,\qquad ax_1+b=\sqrt{e_1}-\sqrt{e_2}-\sqrt{e_3}=z_1,$$
$$ax_2+b=\sqrt{e_3}-\sqrt{e_1}-\sqrt{e_2}=z_2,\qquad ax_3+b=\sqrt{e_3}-\sqrt{e_1}-\sqrt{e_2}=z_3,$$

where $e_1$, $e_2$, $e_3$ are the roots of Euler's equation

$$t^3+3Ht^2+\left(3H^2-\frac{a^2I}{4}\right)t-\frac{G^2}{4}=0.$$

Those that are acquainted with the theory of the Galois realms of rationality will observe that if the algebraic number $x_0$ is adjoined to the realm $R$ of the coefficients, and if the norm of the resulting realm $R(x_0)$ is taken, a realm, say $G=R(x_0)\cdot R(x_1)\cdot R(x_2)\cdot R(x_3)=R(x_0,x_1,x_2,x_3)=R(x_0,x_1,x_2)$ is formed, which is of degree 4!. This realm contains all rational functions of $\sqrt{e_1}$, $\sqrt{e_2}$, $\sqrt{e_3}$ and consequently also all rational functions of $e_1$, $e_2$, $e_3$. Hence $G$ contains as a divisor the realm $A$, where $A$ is the norm of the conjugate realms $R(e_1)$, $R(e_2)$, $R(e_3)$, a realm which is of degree 3!. It follows that the realm $G$ taken with respect to the realm $A$ is of the fourth degree, and that $x_0$ is a primitive quantity in $G/A$. Proofs of these statements are given below without using the Galois theory.

The following results may be established at once: Write $\sigma=z_0=\sqrt{e_1}+\sqrt{e_2}+\sqrt{e_3}$ and $\tau=z_1=\sqrt{e_1}-\sqrt{e_2}-\sqrt{e_3}$. It is seen that $\sigma^2=e_1+e_2+e_3+2(\sqrt{e_1}\sqrt{e_2}+\sqrt{e_1}\sqrt{e_3}+\sqrt{e_2}\sqrt{e_3})$ with corresponding values for $\sigma^3$, $\sigma^4$, $\sigma^3\tau$, etc. Noting that $e_1+e_2+e_3=-3H$, and $\sqrt{e_1}\sqrt{e_2}\sqrt{e_3}=-G/2$, we have the *rational relations*

$$\tau^4+\tau^3\sigma+3H(\tau^2+\tau\sigma+6e_1)+G(3\sigma+5\tau)-2e_1(\tau^2+2\tau\sigma-6e_1)=0,$$
$$\sigma^4+\sigma^3\tau+3H(\sigma^2+\sigma\tau+6e_1)+G(3\tau+5\sigma)-2e_1(\sigma^2+2\sigma\tau-6e_1)=0.$$

[1] These formulæ exhibit the rational expressions desired unless $D=0$; if $D=0$, they may be modified without difficulty.

If in this pair of relations we put $\tau = z_3$, then $\sigma = z_3$. Change $e_1$ to $e_2$, then $\tau = z_1$ when $\sigma = z_3$, and further $\tau = z_2$ when $\sigma = z_0$; change $e_1$ to $e_3$, then $\tau = z_1$ when $\sigma = z_2$ and $\tau = z_0$ when $\sigma = z_3$. The above relations may be written in the form

$$-\sigma = \frac{\tau^4 + \tau^2(3H - 2e_1) + 5G\tau + 18He_1 + 12e_1^2}{\tau^3 + \tau(3H - 4e_1) + 3G},$$

or

$$\frac{-\sigma - \tau}{2} = \frac{e_1\tau^3 + G\tau + 9He_1 + 6e_1^2}{\tau^3 + \tau(3H - 4e_1) + 3G} = A_0\tau^3 + A_1\tau^2 + A_2\tau + A_3.$$

To determine the coefficients $A_0$, $A_1$, $A_2$, $A_3$, interchange $\sigma$ and $\tau$, and we have

$$\frac{-\tau - \sigma}{2} = A_0\sigma^3 + A_1\sigma^2 + A_2\sigma + A_3.$$

Through subtraction, it follows that $0 = A_0(\sigma^3 - \tau^3) + A_1(\sigma^2 - \tau^2) + A_2(\sigma - \tau)$, or

$$0 = A_0(\sigma^2 + \sigma\tau + \tau^2) + A_1(\sigma + \tau) + A_2.$$

Writing for $\sigma$ and $\tau$ their values it is seen that $0 = A_0(-3H + 2e_1 + 2\sqrt{e_2}\sqrt{e_3}) + A_1 2\sqrt{e_1} + A_3$. From this it follows that

$$A_1e_1 = A_0\frac{G}{2} \quad \text{and} \quad A_2 = A_0(3H - 2e_1).$$

These values substituted in the expression above give

$$-\sqrt{e_1} = A_0\left(\sigma^3 + \frac{G}{2e_1}\sigma^2 + 3H\sigma - 2e_1\sigma\right) + A_3.$$

Writing for $\sigma$ its value, it follows, if we put $D = 8e_1^3 + 12He_1^2 + G^2$, that

$$A_0 = \frac{2e_1^2}{D}, \quad A_1 = \frac{Ge_1}{D}, \quad A_2 = \frac{6He_1^2 - 4e_1^3}{D}, \quad A_3 = \frac{3Ge_1(2e_1 + H)}{D}.$$

It is thus seen that

$$\frac{-\sigma - \tau}{2} = \frac{e_1\tau^3 + G\tau + 9He_1 + 6e_1^2}{\tau^3 + \tau(3H - 4e_1) + 3G}$$

$$= \frac{2e_1^2\tau^3 + Ge_1\tau^2 + (6He_1^2 - 4e_1^3)\tau + 3Ge_1(2e_1 + H)}{8e_1^3 + 12He_1^2 + G^2}.$$

We thus have the pair of *integral* relations

$$0 = \sigma(8e_1^3 + 12He_1^2 + G^2) + 4e_1^2\tau^3 + 2Ge_1\tau^2 + (24He_1^2 + G^2)\tau + 6Ge_1(2e_1 + H),$$

$$0 = \tau(8e_1^3 + 12He_1^2 + G^2) + 4e_1^2\sigma^3 + 2Ge_1\sigma^2 + (24He_1^2 + G^2)\sigma + 6Ge_1(2e_1 + H),$$

with five other pairs of relations which correspond to those which have already been indicated for the rational relations.

The completion of the reply to Part 5 will be postponed until after the consideration of Part 6.

6. Writing $x = 2t$, $\lambda = 12H^2 - a^2I$, the reducing cubic is $F(x) = x^3 + 6Hx^2 + \lambda x - 2G^2$. Further put $F_1(x) = x^3 + 3Hx^2 + G^2$. By the algorithm of the greatest common divisor, the remainder, say $F_2(x)$ which is had by dividing $F(x)$ by $F_1(x)$ is $F_2(x) = 3Hx^2 + \lambda x - 3G^2$. The remainder derived by dividing $F_1(x)$ by $F_2(x)$ is, say, $F_3(x) = Ax + B$, where

$$A = \frac{9HG^2 - 9\lambda H^2 + \lambda^2}{9H^2} = \frac{9HG^2 + \lambda(3H^2 - a^2I)}{9H^2},$$

and

$$B = \frac{12G^2H^2 - \lambda G^2}{3H^2} = \frac{G^2a^2I}{3H^2}.$$

The remainder obtained on dividing $F_2(x)$ by $F_3(x)$, expressed also in terms of $F(x)$ and $F_1(x)$ is

$$9H^2[3G^2A^2 + (\lambda A - 3BH)B] = F_1(x)[9H^2A^2 + (3HAx + \lambda A - 3BH)(3Hx + 18H^2 - \lambda)] + F(x)\Phi(x),$$

where as seen below it is unnecessary to write down the value of the polynomial $\Phi(x)$. The term

on the left-hand side of this equation is $G^2a^6(27J^2 - I^3) = G^2a^6\Delta$, where $\Delta$ is the discriminant of the biquadratic under discussion. This proves the theorem proposed. It is of course understood that $G^2$ is *different* from zero.

To continue the discussion further under the preceding Question, write the right-hand side of the above expression in the form

$$F_1(x)[C_0x^2 + C_1x + C_2] + F(x)\Phi(x).$$

It may be shown that

$$C_0 = 9HG^2 - 9\lambda H^2 + \lambda^3 = 9HG^2 + (3H^2 - a^2I)\lambda = (a^2I)^2 - 6H^2a^2I - 9Ha^3J,$$

$$C_1 = 3[3a^3Ja^2I + 2G^2a^2I - H(a^2I)^2 - 18H^2a^3J]$$

$$= 3[6H^2G^2 + \lambda(G^2 + 2H\lambda - 18H^3)]$$

$$= 3[a^3Ja^2I + H(a^2I)^2 - 8H^3a^2I - 18H^2a^3J],$$

$$C_2 = 9(G^4 - \lambda^2H^2) + \lambda^3 + 12G^2\lambda H - 18HG^2a^2I$$

$$= a^6\Delta + 6G^2(3a^3J + Ha^2I).$$

Writing $x = 2e_1$ in the expression above and noting that $F(2e_1) = 0$, it is seen that

$$\frac{1}{8e_1^3 + 12He_1^2 + G^2} = \frac{4C_0e_1^2 + 2C_1e_1 + C_2}{G^2a^6\Delta}.$$

Returning to the integral relations in the preceding Question it is seen that

$$G^2a^6\Delta\tau = B_0\sigma^3 + B_1\sigma^2 + B_2\sigma + B_3,$$

$$G^2a^6\Delta\sigma = B_0\tau^3 + B_1\tau^2 + B_2\tau + B_3,$$

where the $B$'s are integral functions of the second degree in $e_1$. The coefficients of these latter functions are integral in $H$, $G^2$ and $I$. Writing for $\sigma, \tau$ the values $z_0, z_1, z_2, z_3$ and interchanging $e_1$ with $e_2$ and $e_3$, we have here six pairs of integral relations corresponding to the rational relations indicated in 5.

The last part of 5 follows from the first part by use of 4.

NOTE. To Question 36, proposed by Professor Hancock, his own reply, printed above, is the only one as yet received. A reply to part 1 of the question will be welcome.—EDITOR.

## DISCUSSIONS.

We present this month two discussions, both related more closely to secondary than to collegiate mathematics, but of considerable interest also to teachers of the latter. Professor Johnson, in advocating, with much reason, the early use of the complex number and its representation by a point of the plane, has made several comments on which there may well be differences of opinion. It is hoped that expressions on such questions will be forthcoming from readers.

Professor McClenon states without argument certain modes of approaching the teaching of logarithms. Many, perhaps most, teachers will be able to express a preference regarding these diverse methods, without hesitation.

### I. THE COMPLEX QUANTITY IN ELEMENTARY ALGEBRA.

BY W. WOOLSEY JOHNSON, U. S. Naval Academy.

It is the purpose of this note to advocate an earlier introduction, to the student of elementary algebra, of the geometrical interpretation of the complex quantity $a + bi$ (Argand's diagram) than seems usually to be made.

Immediately after the solution of quadratic equations, let it be pointed out that the case of so-called impossible or imaginary roots is but a fresh instance of the occurrence of apparent "anomalies" in the results of mathematical operations, of which several are already familiar to the student. These anomalies always deny the existence of an answer to certain questions in the field of quantity to which they apply. Thus the subtraction indicated by 7–10 denotes the impossibility of an answer in certain number-systems; but in others, notably in connection with the location of a point on a line (a zero point and a unit of length being assumed), such a result presents no difficulty, but gives rise to the so-called negative quantities, of which the very name embodies the earlier view *denying* their existence.

So too we speak of a number (that is, an *integer*) as one that can or cannot be divided by a given smaller number. But when dealing with a straight line all lengths are divisible, and accordingly our number system is enlarged to include fractions.

Again, the indirect operation of extracting a root gave rise to an extension of the field of number, and the name *surd* attached to such a result as $\sqrt{2}$, for instance, reminds us that the attempt to represent it in the field of rational quantity, at which we had hitherto arrived, resulted in a "reductio ad absurdum" proving its impossibility. Yet, by a geometrical construction, the square root of two can be determined with as much exactness as that of an "exact" square.

The new anomaly at which we are supposing the student to have now arrived merits, to be sure, a name implying the impossibility of an answer far more than does, for instance, the negative quantity, which, as a result, may indicate a loss where a gain was expected, or a position on the left where one on the right was assumed. The impossible or imaginary quantity of quadratic algebra on the other hand generally indicates the non-existence of that which was sought, as when we seek to divide a number into parts which shall have a given product. On generalizing the constants, we here find occasion for an interesting class of questions which concern limits of existence and non-existence, taking the form of problems of maxima and minima.

But the question remains, can we find a field of quantity in which questions resulting in quadratic equations can always find an answer?

Having denoted the roots of $x^2 + 1 = 0$ (the simplest quadratic which presents the anomaly) by $\pm i$, we find the roots of every quadratic to be expressible in one of the forms $a \pm b$ or $a \pm bi$, in which $a$ and $b$, being quantities of the kind hitherto known, may be called "scalars," because measurable on a scale or straight line. But, whereas in the first case the parts may be merged into single scalars, they are in the second case *heterogeneous* and cannot be so merged. Space of two dimensions now presents us the opportunity to form another scale, just as it does when we seek to represent a function of a scalar associated with its independent variable. But whereas, in that case, we were free to choose the direction of the new scale, consideration of $i$ as a turning factor fixes the direction of the scale of imaginaries. It is, in my opinion, not necessary to say that we have

*invented* the new numbers $a + bi$ and annexed them to complete our field of number; rather let us say that, as in the former case of the straight line for scalars, we have found, ready to hand, a field of quantity of the desired kind, namely, one[1] in which the roots of a quadratic are always possible.

The mode of performing geometrically the fundamental operations of algebra with complex quantities; the mode in which the idea of absolute magnitude is attached to them; the construction of the roots of unity with their introduction to the many-valued function; and later their application to the power series; can hardly fail to interest and stimulate the student.[2]

Finally we observe that, at this point, the student will have an intelligent appreciation of the meaning of the fact, which may now be stated (without proof, for which he is, of course, not yet prepared), that every equation of the $n$th degree has $n$ roots; no new anomaly appearing in the solution of those of higher degree, although new incommensurables do occur; thus the system of complex quantities is a complete set reproducing itself through all the operations of algebra, direct and indirect.

## II. On the Teaching of Logarithms.

By R. B. McClenon, Grinnell College.

How should the subject of logarithms be presented to the beginner? The usual method has been, to start with the laws of exponents, define logarithms as powers of the base, and thus infer the fundamental laws of operation. This approach, however, is extremely likely to prove confusing to the student, because it does not bring out clearly enough the functional relation involved; and this disadvantage is accentuated by the impossibility of giving a satisfactory discussion of the meaning of an irrational exponent until a later stage of the mathematics course.

Two other methods have been proposed as a substitute for this one. One suggestion is, to use the historical approach, as employed by Napier in his invention of logarithms.[3] This would begin with the fundamental idea of the relation between an arithmetic and a geometric progression, the notion of "base" being at first entirely ignored. With the help of graphical methods, and an abundance of concrete problems chosen from physics, economics, and finance, it is said to be possible to bring out clearly and vividly the relation between the exponential and logarithmic functions. The other suggestion is, to abandon frankly all attempt to *explain* logarithms at the outset, but rather to focus the attention upon the actual use of logarithmic tables, until the mechanical rules for computa-

[1] A former student of Kronecker once told me that Kronecker would never admit the existence of a complex quantity. He said "there was not any complex variable $x + iy$. There were two variables, $x$ and $y$." Whatever we say to this verbal contention, we shall continue to say that it takes two quantities (*i. e.*, scalars) to determine the position of a point in a given plane.

[2] In this connection the writer may refer to a construction of the roots of the quadratic with complex coefficients given by him to the New York Mathematical Society in 1893 and, in abstract, in the *Bulletin*, Vol. 2, p. 171.

[3] See, for example, Cajori, "History of the Exponential and Logarithmic Concepts," Am. Math. Monthly, Vol. 20 (1913), pp. 5–14.

tion have become thoroughly familiar. Then, it is urged, the student will find it easier to appreciate the theoretical discussion which comes later, as this will merely take the line of explaining the reasons for facts of whose truth he is already convinced.

Both of these suggestions seem worthy of being tried out, and they may very likely have been already. If so, it is to be wished that some testimony as to the results might be presented in this department of the MONTHLY; as well as any other suggestions that have been found helpful in the teaching of this subject.

The last plan mentioned above, the postponement of the theoretical part of the work until the practical use of logarithms has been mastered, would prove most satisfactory if this practical part of the work could be introduced into the secondary school mathematics course, as would certainly seem desirable on many grounds. When this is done, the problem of the college teacher will be much simplified, and, what is of course of incomparably greater importance, the secondary school student will have the added knowledge and power that come from acquaintance with the practical use of so valuable a mathematical tool.

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## RECENT PUBLICATIONS.

### REVIEWS.

### HINDU SCIENCE.

*Hindu Achievements in Exact Science, A Study in the History of Scientific Development.* By B. K. SARKAR. Longmans, New York, 1918. 13 + 82 pp. Price, $1.00.

"The Astronomical Observatories of Jai Singh." By G. R. KAYE. *Archeological Survey of India,* New Imperial Series, Vol. 40. Calcutta, 1918, 8 + 154 pages + 26 plates and frontispiece photograph of Jai Singh. Price 15 rupees.

These two works are illustrative of two well-defined tendencies which have been existent for fifty years or more, in the discussion of Hindu science. The work of Sarkar is that of the enthusiast for all things Hindu; here we have the acceptance of practically all claims for the Hindu origin of different scientific ideas, with the general denial of foreign influence. In the work of Kaye we have the insistence upon the absolute dominance of foreign ideas in Hindu science, with the practical denial of any indigenous contribution.

Unfortunately *Hindu Achievements* is written by one who does not treat historical material critically; thus we have works of real value and works of no value cited as authorities, with a preponderance in favor of obsolete and even worthless works relating in one way and another to Hindu science. On the other hand Mr. Kaye is familiar with the modern authorities in the field of the history of science. This work of Jai Singh is a work of real merit but even it exhibits the tendency to depreciate the value of Hindu contributions which has characterized much of Mr. Kaye's work.

The denial of Hindu originality in science is paralleled by the denial on the part of certain writers of Babylonian, Egyptian, and even Arabic originality. This denial rests upon a fundamental misconception of the nature of science and scientific progress; it is of the same naïve nature as the common German view (before the war) that all discoveries in science of any value were German in origin.

To deny to Babylon, to Egypt, and to India, their part in the development of science and scientific thinking is to defy the testimony of the ancients,[1] supported by the discoveries of the modern authorities.[2] The efforts which have been made to ascribe to Greek influence the science of Egypt, of later Babylon, of India, and that of the Arabs do not add to the glory that was Greece. How could the Babylonians of the golden age of Greece or the Hindus, a little later, have taken over the developments of Greek astronomy? This would only have been possible if they had arrived at a state of development in astronomy which would have enabled them properly to estimate and appreciate the work which was to be absorbed. People in one stage of civilization do not borrow the science of another people in a higher state of civilization. There has never been any question concerning the nature and origin of such feeble beginnings of science as are found among the American Indians. With the Hindus and the Babylonians and the civilization of Europe in the time of Alexander the Great and up to 600 A. D., the problem is entirely different. Here we have peoples who had reached approximately the same stage of development. The admission that the Greek astronomy immediately affected the astronomical theories of India carries with it the implication that this science had attained somewhat the same level in India as in Greece. Without serious questioning we may assume that a fundamental part of the science of Babylon and Egypt and India, developed during the times which we think of as Greek, was indigenous science. Nor do we thereby detract from the real greatness of Greece. The Hellenic civilization remains as an integral and vital part of all civilization, and not as something apart.

Recently Mr. Kaye has published an article[3] in which the denial of Hindu originality has reached the ultimate limit, including not only all Hindu astronomy, but even the sine function, the Hindu numerals, the value for $\pi$ of 3.1416 and the solutions of indeterminate equations. To justify this total repudiation it is necessary here to postulate the nature of the contents of Greek works which are lost. This is a new method in history, and not one to be commended. Not a scintilla of real historical evidence is advanced to support the contentions. Those who lack training in the weighing of historical evidence may take a new *theory* for *proof*, but there is a profound difference.

[1] Hipparchus and Ptolemy, in Ptolemy's *Almagest*; Theon of Smyrna, and Diodorus Siculus; Herodotus, II, 109; Berossus, as given in fragments; Clemens of Alexandria; Pliny, *Hist. nat.*, VI, 122 and VII, 193; and numerous others.

[2] Heath, *Aristarchos of Samos* (Oxford, 1913); Berthelot, *Les origines de l'alchimie* (Paris, 1885), Chap. III; Boll, *Sphaera* (Leipzig, 1903), *p.* 461; Cumont, *Astrology and Religion among the Greeks and Romans*; Kugler, *Die babylonische Mondrechnung* (Freiburg, 1900), pp. 50–51, 200–211; Epping, *Astronomisches aus Babylon* (Freiburg, 1889), pp. 183–190.

[3] G. R. Kaye, "Influence grecque dans le développement des mathématiques hindoues," *Scientia*, Vol. 25 (Jan., 1919), pp. 1–14.

The work of Mr. Sarkar is in no sense scholarly. As a popular exposition it may be of interest particularly to Hindus, but it is distinctly unfortunate that so much is claimed which is easy to refute. This only results in the non-acceptance even of those claims which are justified.

Kaye's work shows the truly magnificent equipment of the observatories established by Jai Singh (1686–1743). Only among the Arabs would we find any buildings which could be compared with the Delhi, Jaipur, Ujjain, Benares, and Mathura structures. At Delhi quadrants, in the plane of the equator, with a radius of 49.5 feet, are graduated to minutes; the high masonry gnomon, 7.5 feet in width and 113.5 feet in length, rises above these quadrants to a height of about 68 feet, and its shadow on the quadrants gives the apparent solar time. Two complementary concave hemispheres of masonry work, diameter 27 feet, 5 inches, formed a part of the equipment; so also did two circular buildings with inside diameter of 49.1 feet, with 30 openings each 6° in width. Other masonry instruments and metal astrolabes and armillary spheres, in large part of Persian and Arabic workmanship, still survive from the instruments which were used in these observatories. A set of astronomical tables was prepared by Jai Singh, dedicated to the Emperor Muhammad Shah, being called Zij Muhammad Shahi; these tables follow closely the methods of Ulugh Beg. However, "polar longitudes" are given, as in the Surya Siddhanta, and it is more reasonable to suppose a Hindu source than a Moslem one, as Kaye does.

The wonderful ruins of these observatories of Jai Singh are a tribute to Hindu science, revealing an unusual power of appreciation of pure science whether or not they bear direct testimony to other positive achievements of the Hindus themselves.

LOUIS C. KARPINSKI.

*The Italian Universities and their Opportunities for Foreign Students.* By KENNETH MCKENZIE. Roma, Tipografia Nazionale Bertero, 1919. 8vo. 16 pp.

This timely and very interesting pamphlet was written by Dr. McKenzie, professor of romance languages in the University of Illinois, but now Director of the Italian branch of the American University Union in Europe. The headings of the thirteen sections are as follows—'The Universities of Italy: their history; The universities of Italy: their present condition; Primary and secondary schools; Organization of the universities;—["The university year extends nominally from October 16 to July 31; but the instruction begins in November or even later, and ends by June 15. Examinations are as a rule given in two periods, beginning respectively October 16 and June 16"]—Registration and fees; University courses and degrees;—["The University courses are of four years in the case of the Faculties of Letters and Philosophy, Sciences and Law"]—Foreign students;—["Foreigners who have received a degree from a University of approved standing, and who can prove that they have studied all the subjects required for one of the regular degrees in Italy, are not obliged to study in residence or to take the examinations in the separate subjects, but may present themselves directly for the final examination for the degree which they desire. This does not apply, however, to the new degree mentioned below. The character of the final examination is . . . written dissertation and oral discussions."]—Requirements for the regular degrees; The new doctor's degree;—["On October 28, 1917, a new doctorate degree was established, equivalent in grade and in the amount of work required to the old four years' degrees, but not demanding their rigid selection of courses. This new degree is not professional in the sense of giving legal rights in connection with the practice of the professions in Italy, but is intended to encourage scientific attainment for the purpose of advancing knowledge and increasing

the personal culture of the individual; for this reason it is sometimes referred to as a scientific degree (as opposed to professional). It was destined at first exclusively for foreigners, who would not wish to conform to the requirements of the professional curricula. In March, 1918, however, it was opened to Italians as well."]—Degrees for foreigners;—["At present no degree of higher grade than the regular laurea or the new doctorate is conferred in Italy; and from what precedes, it is obvious that the requirements for these degrees are less in amount than the requirements for the doctorate in Graduate Schools of good standing in America. It must be borne in mind, however, that for Americans the successful following of a course of university study in Italy implies residence abroad and mastery of the Italian language"]—The language; Choosing a University; —["Concise descriptions of all the Universities and other higher institutions in Italy, including libraries and learned societies, with lists of the professors and other officials, will be found in the *Annuario degli Istituti Scientifici Italiani*, compiled by Professor Silvio Pivano for the Associazione Italiana per l'Intesa Intellettuale fra i Paesi Alleati ed Amici (Rome, 1918; price 10 lire)."]—Special provisions 1919–1920.

*Number Stories of Long Ago.* By D. E. SMITH. Boston, Ginn, 1919. 12mo. 7 + 136 pp. + 8 plates in color. Price 48 cents.

Extract from "Preface number two for the grown-ups, and not worth reading"—

". . . This book is intended for supplementary reading in the elementary school. It is written in nontechnical language, and the effort has been made to connect with the history enough of the human element to make it more interesting than any mere recital of facts. With it there is also joined something of the history of writing materials, this being connected naturally with the story of our numbers. Chapters I–VIII can easily be read aloud, and the Question Box at the end of each chapter can be used as a basis for conversation or for written work.

"The facts stated in the book are as nearly exact as the circumstances permit. It is not to be expected, however, that changes in the form of various numerals will be considered. Such changes are of no moment in a work of this nature and do not contradict the statement that the historical facts are presented with substantial accuracy.

"It is the author's hope that this little series of human incidents will create a new interest not merely in the study of arithmetic but in the story of the development of our civilization."

The solutions of the problems in chapters IX and X of *Number Stories* have been given by D. E. SMITH in a fourteen page pamphlet entitled: *Number Puzzles before the Log Fire.*

*Essentials of Algebra and Geometry.* By F. M. MORGAN. New York, Association Press, 1919. 12mo. 58 pp.

This is the second of the series, published under the direction of Professor J. W. Young for the National War Work Council of Young Men's Christian Associations, to which reference has been made already in the MONTHLY (March, 1919). It contains six lessons preparatory to the study of trigonometry. The requirement has been cut down to a minimum and it is intended that everything given shall be of importance. There are numerous "Exercises," "Oral review exercises" and "Review exercises."

## NOTES.

The *Harvard Alumni Bulletin* for April 24, 1919, contains the report by a Faculty Committee (of which Professor G. D. BIRKHOFF is a member) on General Final Examinations for Degrees. Such examinations are not to be used in the Divisions of Mathematics and the Natural Sciences.

The twenty-page *List of Members, 1919*, of the Indian Mathematical Society contains 195 names. Of these one name is that of a "Patron," one of an "Honorary Member," and nine of "Life Members." Of the 185 "Ordinary Members" only one resides outside of India.

The thoroughly unsound foundation for J. M. Child's contention that "Isaac Barrow was the first inventor of the infinitesimal calculus" was set forth clearly by Professor Cajori in this MONTHLY for January, 1919. It seems unfortunate, therefore, that the contention is whole-heartedly accepted by such an influential journal as *Science Progress*. (Cf. the issue for April, 1919.)

Announcements of the Cambridge University Press: Sir Thomas L. Heath's *Euclid in Greek* (Book 1) will include the Greek text of Euclid's first book together with an English translation and notes; the author holding the view that neither Euclid nor Greek can ever be more than apparently in abeyance—New and completely revised editions of Love's *Elasticity* and Lamb's *Infinitesimal Calculus* are in the press—Professor J. H. Jeans has written a new work entitled *Problems of Cosmogony and Stellar Dynamics*, which it is hoped to publish this autumn.

*Revista de la Sociedad Matemática Española* ceased publication with the completion of año 6 in July, 1917. In January, 1919, a new mathematical periodical, entitled *Revista matematica Hispano-Americana*, appeared at Madrid under the editorship of J. Rey Pastor. It is the Sociedad's official organ. At this writing five numbers have been received in America. Their contents include: "Notas sobre la teoria de grupos"[1] by G. A. Miller (pages 148–152), a sketch and fine portrait of Don Eduardo Torroja, 1847–1918, (pages 1–13), and a portrait and sketch of Hadamard and his scientific work (pages 65–80, 105–112). In the latter is given a list of 135 papers published by Hadamard 1888–1914.

We have also received Tome 1, no. 1 of Professor Z. G. de Galdeano's new periodical: *Supplemento a la Revista matemática Hispano-Americana, boletin, de critica, pedagogia, historia y bibliografia* (32 pages). Professor Galdeano has also written or edited four other periodicals: (1) *El Progreso Matematico*, 7 vols., 1891–95, 1899–1900; (2) *Boletin de critica, ensenanza, y bibliografia*, 2 nos., 1907–08; (3) *Supplemento a la revista de la Sociedod Matematica Española*, 3 nos., 1917; (4) *El progreso scientifico, revista semestral*, 1 no., July 1918 (See this MONTHLY, March, 1919, p. 118).

L. Huxley has recently published his *Life and Letters of Sir Joseph Dalton Hooker*,[2] the distinguished botanist and traveller who died in 1911 at the age of 94. The first volume contains interesting particulars (pages 538–546) of the famous $x$ Club[3] of which there were 240 meetings from the first in 1864 to the last in 1892. No additions were ever made to the original membership of nine: Hooker, T. H. Huxley, John Tyndall, Herbert Spencer, Edward Frankland, John Lubbock, George Busk, and the mathematicians William Spottiswoode and T. A. Hirst. The first break in the circle was caused by the death of Spottiswoode in 1883.

[1] In part, a translation of "Remarks on the bearing of the theory of groups" in *Tôhoku Mathematical Journal*, vol. 11, 73–78, December, 1914.

[2] New York, Appleton, 1918.

[3] See also *Life and Letters of T. H. Huxley*, Vol. 1, 368 *seq.*; *Sketches from the Life of Edward Frankland*, page 148 *seq.*; and Huxley's reminiscences of John Tyndall in the *Nineteenth Century*, Jan., 1894.

There was only one meeting of the Club after Hirst's death in February, 1892.

The second volume of the *Life* contains the following extracts (pages 126, 336) culled from Hooker's letters:

(1) To *Charles Darwin*, August 5, 1871: "I have been reading W. Thomson's[1] address [at the Edinburgh meeting of the British Association for the Advancement of Science in 1871], and am anxious to hear your opinion of it. What a belly-full it is, and how Scotchy! It seems to be very able indeed, and what a good notion it gives of the gigantic achievements of mathematicians and physicists—it really makes one giddy to read of them. I do not think that Huxley will thank him for his reference to him as a positive unbeliever in spontaneous generation—these mathematicians do not seem to me to distinguish between un-belief and a-belief—I know no other name for the state of mind that is traduced under the term scepticism. I had no idea before that pure mathematics had achieved such wonders in practical science, and I wonder how far Thomson's statements will be contested. The total absence of any allusion to Tyndall's labors, even when comets are his theme, seems strange to me."

(2) To *Rev. J. D. La Touche*, Dec. 24, 1893: "What you say of A, B, and C does not surprise me. They are 'ne plus ultra' mathematicians, have not a conception of biological science, and in fact are only *half intellects* (I suppose I deserve to be burned), but so it is, that I have often found such men to be impervious to reasoning out of their own circle, in matters of natural science. With biologists, who have to found everything, beyond pure observation, on circumstantial evidence, the case is quite different. For hundreds of biologists who are good mathematicians, you will not find ten vice versa."

## ARTICLES IN CURRENT PERIODICALS.

**BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY**, volume 25, no. 7, April, 1919: "Mathematics in war perspective" by L. E. Dickson, 289–311 [Presidential address delivered before the American Mathematical Society, December 27, 1918]; "A partial isomorph of trigonometry" by E. T. Bell, 311–321; "The trains for the 36 groupless triad systems on 15 elements" by Louise D. Cummings, 321–324; "A theorem on areas" by T. Hayashi, 324–325; "Concerning the definition of a simple continuous arc" by G. H. Hallett, Jr., 325–326; "The transformation of a regular group into its conjoint" by J. E. McAtee, 326–329; "Notes" and "New publications," 329–336.—No. 8, May: "The life and services of Maxime Bôcher" by W. F. Osgood, 337–350; "A theorem on linear point sets" by H. Blumberg, 350–353; "A general form of Green's Theorem," 353–357; "Rotating cylinders and rectilinear vortices" by H. Bateman, 358–374; Review by A. Dresden of Shaw's *Lectures on Philosophy of Mathematics* (Chicago, 1918), 374–377; Review by R. D. Carmichael of Macrobert's *Functions of a complex variable* (London, 1917), 377–378; Review by A. Emch of Montessus de Ballore's *Lecons sur les fonctions elliptiques en vue de leurs applications* (Paris, 1917), 378–379; "Notes" and "New Publications," 379–384.—No. 9, June: "The March meeting of the American Mathematical Society at Chicago," 385–392; "The April meeting of the San Francisco section, 393–397; "On a certain generation of rational circular and isotropic curves" by Arnold Emch, 397–404; "The self-dual plane rational quintic" by L. E. Wear, 405–408; "Groups containing a relatively large number of operators of order two" by G. A. Miller, 408–413; "The derivative of a functional" by P. J. Daniell, 414–416; Review by L. W. Dowling of *Scritti matematici offerti ad Enrico D'Ovidio*, etc. (Torino, 1918), 417–422; "Shorter Notices," 422–424; "Notes," 424–429; "New Publications," 429–432.

**JOURNAL OF ACCOUNTANCY**, New York, volume 27, 1919, January: "Practical interpolation" by A. S. Little, 48–60.—April: "Rapid calculation of compound interest processes" by F. C. Belser, 241–248; "Mathematics of credit extension" by F. Thulin, 259–267.

**JOURNAL OF EDUCATIONAL PSYCHOLOGY**, Hershey, Pa., volume 10, no. 2, February, 1919: "English and mathematical abilities of a group of college students" by E. C. Tolman, 95–103. [Based on observations made on an introductory class in psychology at the University of California made up of sophomores, juniors and seniors (men and women)].

[1] William Thomson (1824–1907), afterwards Lord Kelvin, was the son of James Thomson who became professor of mathematics at the University of Glasgow where William matriculated when he was a little more than ten years of age, and was afterwards to be for fifty-three years a "professor of Natural Philosophy."

**JOURNAL OF PHILOSOPHY, PSYCHOLOGY AND SCIENTIFIC METHODS,** volume 15, no. 10, May 9, 1918: "Doctrinal functions" by C. J. Keyser, 262–266. [Synopsis in *Mind*: Starts from Russell's notion of a propositional function which is neither true nor false until values have been assigned to its variables, and points out that values may always be given which make nonsense of the function and hence are to be called *inadmissible constants*. Admissible constants are divided into *verifiers* and *falsifiers*: the former "*satisfy* it and are called the *values* of its *variables*. Thus the values of a given function are the true propositions that are derivable from it by replacing its variables by admissible constants." Applying these distinctions to "the postulational method of founding and constructing mathematical sciences," it appears that as "any postulate-system contains one or more undefined terms and at least one of them denotes an *element*," which gives it the appearance of having a definite subject-matter, the system will require interpretation. In this process "the *rôle* of the undefined terms is the *rôle* of variables"; hence "a postulate system is not a system of propositions, as it is commonly said to be, but it is a system of propositional functions." It should be called therefore 'a doctrinal function,' and it is shown that "the number of values of any doctrinal function is equal to any given transfinite cardinal number. It is a corollary that "Hilbert's *Foundations of Geometry* is not a geometry at all, nor is it any other doctrine; it is a doctrinal function having an infinitude of values, some of them geometric, some of them algebraic, some of them neither one nor the other."]—No. 11, May 23: "The definition of infinity" by R. H. Dotterer, 294–301. [Synopsis in *Mind*: "Criticises the new 'infinity' of Dedekind and Cantor as doubly ambiguous. (1) Two infinite series do not stand *merely* in a one-to-one correspondence, but also in an infinity of others. But *unless* they do, the new definitions of 'similarity' and 'equality' break down. (2) The 'new infinite' is only the old in disguise, for that also involved an inexhaustible series and the possibility of a one-to-one correspondence (or of any other). Hence it retains also the old difficulties. Only they are hidden away in its definition. Thus the infinite series of cardinal numbers cannot be called a 'system' or a 'totality' without assuming a realised infinite. If 'totality' is defined to mean *determinable* only, the 'new infinite' cannot claim existence any more than the old. Hence it does not help in the solution of any of the problems of philosophy or theology."]—No. 14, July 4: "Mechanism and causality in physics" by M. P. Cohen, 365–386—No. 21, October 10: "A list of articles, mostly book reviews, contributed by Charles S. Peirce to *The Nation* to which is appended some additions to the bibliography of his published writings in this *Journal*, December 21, 1916," 578–584.

**JOURNAL OF THE INDIAN MATHEMATICAL SOCIETY,** volume 11, February, 1919: Frontispiece, Indian Mathematical Society, 2d Conference, January, 1919, Bombay; "Second Conference," 3–30; "Introductory Notes to differential geometry" by E. H. Neville, 30–33; "On the factorization of two large numbers" by N. B. Mitra, 34; Questions and Solutions, 37–40.

**MATHEMATICAL GAZETTE,** volume 9, March, 1919: "The Annual Meeting of the Mathematical Association—Report of the Council for 1918," 309–311; "Mathematics and the pivotal industries" by W. P. Milne, 312–316; "The teaching of geometry to first-year pupils" by B. A. Howard, 317–321; "Cubic graphs of the form $y = ax^3 + bx^2 + cx + d$" by A. Lodge, 322–326; Review by H. Hilton of Frost's *An Elementary Treatise on Curve Tracing* (London, 1918); [Quotation: "The reviser's hope that 'this edition will be found to be comparatively free from inaccuracies' unfortunately proves fallacious on a casual glance at the diagrams. Some errors, e. g. plate V., fig. 24, and VI. 17, are doubtless due to the printer. But III. 27; IV. 25, 26, 29 represent curves cut by a line in points whose number is greater than the curve's degree; while IV. 28 is also rather rough and ready. There is a tendency to draw cusps as though the tangents thereat were distinct, to exaggerate the rate of approach of a curve to its asymptote, and to misplace inflexions and their tangents. This is the more to be regretted, in that these are just the mistakes against which a beginner most needs warning"]. Review by W. P. Milne of Carey's *Infinitesimal Calculus* (London, 1917–18), 327–328; Review of Whitehead's *The Organization of Thought* (London, 1917), and Hermite's *Oeuvres*, tome 4, (Paris, 1917); "The Teacher's Library, "Section II, by W. P. Milne, i–v.

**MEMOIRS OF THE NATIONAL ACADEMY OF SCIENCES,** Washington, D. C., Volume 14, second memoir, 1919: "Complete classification of the triad systems on fifteen elements" by H. S. White, F. N. Cole, and Louise D. Cummings, 1–89: [Part 1: "Triad systems on 15 elements whose group is of order higher than unity" by H. S. White, 5–25; Part 2: "Trains for triad systems on 15 elements whose group is of order higher than unity" by L. D. Cummings, 27–68 (216 figures); Part 3: "Groupless triad systems on 15 elements" by H. S. White and L. D. Cummings, 69–72; Part 4: "Structure as defined by interlacings, heads, and semiheads; a complete census of triad systems in fifteen elements" by F. N. Cole, 73–80; Part 5: "Sequences and indices for all groupless triad systems on fifteen elements" by L. D. Cummings, 81–89.]

**MONIST,** volume 29, no. 2, April, 1919: "Pandiagonal magics of orders 6 and 10 with minimal numbers" by C. Planck, 307–316; "The publication of *Isis*" by G. Sarton, 318–319—[Reprinted from *Science*, February 14, 1919]; Review of *The philosophy of Mr. B*rtr*nd R*ss*ll, with an appendix of leading passages from certain other works.* Edited by P. E. B. Jourdain (London, Allen and Unwin, 1918), 319–320. [Quotation: ". . . No amount of quotation (finite by reason of editorial control) could exhaust the transfinite number of good things in this amusing volume. For the proof that this is not mere hyperbole I must refer the reader to the chapter on 'The Hierarchy of Jokes.' And when one comes delightedly across such gems of delicate irony as the logical analysis of Mr. Chesterton's method of disguising platitudes as paradoxes (p. 41) one can only hope that Mr. Jourdain will discover among the papers of the late Mr. R*ss*ll more, and still more, pin-prickings of popular bombastics."]

**NIEUW ARCHIEF VOOR WISKUNDE,** Amsterdam, series 2, volume 12, no. 4, 1918: "Nieuwe behandeling van een bekend kansvraagstuk" by F. Schuh, 361–373; "Remarque sur la singularité que présente la courbe nodale d'une surface développable quand l'arête de rebroussement possède une tangente double" by W. A. Versluys, 374–378; "Onderzoek van de functie die voor $R(z) > 0$ wordt voorgesteld door de reeks

$$\sum_{n=1}^{\infty} \left(\frac{n}{a}\right) \frac{e^{-ns}}{n^s}$$

by N. G. W. H. Beeger, 379–392; "Ueber grőszten gemeinsamen Teiler und kleinstes gemeinsames Vielfaches" by I. G. van der Corput, 393–404; "Constructie van de assen der ellips, die de centrale projectie is van een cirkel" by F. Schuh, 405–406; "Beweging van een materieel punt op den bodem eener draaiende vaas onder den invloed der zwaartekracht" by L. E. J. Brouwer, 407–419; "Elementair bewijs der intgebreide wederkeerigheidswet van Legendre" by A. L. Bartelds and F. Schuh, 420–438; "Addenda en corrigenda over de grondslagen der wiskunde" by L. E. J. Brouwer, 439–445; "On cylindrical functions" by L. Crijns, 446–449; "Over het differentieeren van bepaalde integralen" by J. Wolff, 450–453; "Eenige stellingen over limieten van integralen" by H. Bremekamp, 454–460; "Notes on some differential equations in physics and differential geometry" by T. Hayashi, 461–468; Review (in Dutch), 468–480.

**NOTES AND QUERIES,** London, 12th series, volume 5, February, 1919: "EULER ON THE END OF THE WORLD.—Euler the mathematician (1707–83) is said to have predicted that the end of the world would take place in a certain year. It is likely that some reference to the statements would be found in the letters of Catherine II (1729–96) to F. M. Grimm (1723–1807).

"Could a reader give some precise information?

R. G. H."

**NOUVELLES ANNALES DE MATHÉMATIQUES,** volume 77, December, 1918: "Sur les troisième et quatrième centres de courbure des courbes de Cesàro" by R. Goormaghtigh, 441–445; "Sur un problème concernant des groupes de points sur l'hyperbole équilatère" by R. Goormaghtigh, 445–448; "Sur les nombres complexes de deuxième et de troisème espèce" by L. G. Du Pasquier, 448–461; "Propriété de certaines courbes et surfaces enveloppes" by M. Weill, 462–464; "Sur les cercles bitangents a la parabole" by J. Bouchary, 464–471; "Sur les courbes définies par une relation entre les distances de chacune de leurs tangentes à des points fixes" by M. d'Ocagne, 471–472.

**PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY,** series 2, volume 17, part 5, March, 1919: "Professor Olaus Henrici" by M. J. M. Hill, xlii–xlix. [Quotations: "Olaus Magnus Friedrich Erdmann Henrici was born in the year 1840, in Meldorf, on the West Coast of Holstein, where his father held a post in the Danish Civil Service. His father had studied Science and Engineering at the Gewerbschule in Berlin, which afterwards became the College at Charlottenburg, and he was one of the three who prepared the plans for the canal from Kiel to the mouth of the Elbe . . . Henrici will be remembered chiefly as a great teacher. During his years of struggle [1865–1869; he went to London in 1865] when he was obliged to spend much time in giving lessons in elementary mathematics to school boys, he had learned to probe the working of the minds of his pupils. This was of great value to him in his subsequent career as a teacher and a writer, for he acquired the faculty of expressing himself with great clearness in both capacities. He published nothing until he felt completely satisfied as to its form. But for this characteristic more of his methods and ideas would have been preserved. I understand that he has left a large amount of manuscript, and it is much to be hoped that some one will be found to go through it with extreme care."]

**SCHOOL SCIENCE AND MATHEMATICS,** volume 19, no. 5, May, 1919: "Construction work in solid geometry" by E. W. Schreiber, 407–413; "Historical notes in the mathematical text-books" by G. A. Miller, 414–416; "Graphical algebra" by F. E. Nipher, 417–420; "History of mathematics at University of California" by O. Schmiedel, 462; [Quotation: "Three courses are given the first term of the present session; a largely attended course on the History of Mathematics and Physics, a course on The Teaching of Mathematics in Secondary Schools, and a most interesting seminar course on The History of Fundamental Concepts of the Calculus and of Fluxions."] "Useful benefits from study of mathematical history" by O. Schmiedel, 463; Problems and solutions, 468–473.

**SCIENCE PROGRESS,** volume 13, No. 3, January, 1919: Mr. Oscar S. Adams has called attention to the fact that, in quoting from R. Ross's article on isosceles trigonometry in our April issue, two misprints occur: (1) *for* "(bas $s\theta + i$ bas $\theta)^n =$ bas $sn\theta + i$ bas $n\theta$" *read*

$$\text{"(bas } s\theta + i \text{ bas } \theta)^n = 2^{n-1}(\text{bas } sn\theta + i \text{ bas } n\theta)\text{"};$$

(2) *for* "between $\theta$ and $\pi$" *read* "between 0 and $\pi$." The first of these slips was due to an error in the original. Mr. Adams remarks further: "De Moivre's theorem can of course apply only to complex numbers that lie on the unit circle with the origin as center or, in other words, to those with modulus equal to unity. bas $s\theta + i$ bas $\theta$ lies on the circle with radius equal to two"—No. 4, April: "The discovery of the Calculus" by R. R., 634–635; "Mathematics in an encyclopædia" by P. E. B. Jourdain, 648–651; [First paragraph: "A discussion of the nature and extent of articles in a proposed mathematical dictionary has lately excited a good deal of interest in America. The Mathematical Association of America has appointed a committee to investigate the subject, and Dr. G. A. Miller (AMER. MATH. MONTHLY, 1918, *25*, 383–7; cf. 428) has given an example of a proposed article dealing with the theory of groups. This article has the conventional characteristics of an article for a dictionary. This is not meant to imply that the article is not a good one of its kind: it is, in fact, a very thorough and exhaustive treatment of the things about which it means to give information. But it seems to me that what it and most other dictionary articles discuss is exactly what nobody really wants who is not bent on acquiring merely the sort of knowledge that is required by examinations. Nearly all of Dr. Miller's specimen article is devoted to definitions of "groups" in the general technical meaning of the word, and particular qualities of groups. It is no uncommon thing to read in an examination paper such a question as: 'Define the terms *isomorphism* and *transitivity*;' but such information, at least in this form, is not required outside the examination room. Surely an intelligent being who consults an article on a science in an encyclopædia does so from a wish to know what such-and-such a department of knowledge is about and what has been done in it, whether he does so for cultivating his own mind, or as a preliminary for cultivating the minds of others, or for his own work of discovery. He does not want simply to fill his memory with what certain words mean in the technical language of the present: his aim would be to get a firm grasp of the principles underlying that particular subject, and to find out why certain notions were so important as to be fixed by a name—such as "isomorphism," for instance. It is not the formal *definitions* that we must seek in the first place; it is those more or less vague ideas which have lost part of their vagueness so as to become apparently definable. Of course we can never be quite certain that the definitions we may arrive at in some science at a particular time are quite free from vagueness: a mathematician who is interested in the principles of his subject can find many instances of 'definitions' which were seriously given only a few years ago and which can now be seen not to define at all. The theory that definitions should form the subject-matter of articles in a dictionary or encyclopædia, and a great part of the subject matter of text-books, is a theory held by those schoolmasters who think that examinations are the goal of knowledge; and we must always remember that professors are only a better class of schoolmasters." In the remainder of the article Mr. Jourdain deals chiefly with the mathematical articles in the *Encyclopædia Britannica.*] "The general theory of relativity and Einstein's theory of gravitation" by G. W. Tunzelman, 652–657—[Reprinted in *Scientific American Supplement*, May 31, 1919, Vol 87, pp. 346–347]; Reviews by P. E. B. Jourdain of Bateman's *Differential equations* (London, 1918), Veblen and Young's *Projective Geometry* (Boston, 1910–1918), Vollenhoven's *De Wijsbegeerte der Wiskunde van Thëistisch Standpunt* (Amsterdam, 1918) and Tuttle's *The Theory of Measurements* (Philadelphia, 1916), 666–672; Review by A. E. Heath of Jourdain's *The Philosophy of Mr. B*rtr*nd R*ss*ll with an appendix of leading passages from certain other works* (London, 1918), 670–671; Review by H. S. J[ones] of Kaye's *The Astronomical Observatories of Jai Singh*

(Calcutta, 1918), 672–674; Review by E. H. B. of C. V. Raman's *On the mechanical theory of the vibrations of bowed strings and of musical instruments of the violin family, with experimental verification of the results*, Part I (Calcutta, 1918), 683. [Quotations: "Helmholtz on an experimental basis was able to construct a partial theory of the bowed string. F. Krigar Menzel and A. Raps photographed, upon a revolving drum carrying a film, various points of bowed strings so as to exhibit their displacement-time graphs. E. H. Barton and his pupils took simultaneous photographs of the behaviour of the strings and either bridge, belly or air of a monochord or violin. But in none of the foregoing cases was a direct mechanical theory of the string, bridge, etc., attempted. This is now done by C. V. Raman.

"The equations of motion of the string are written and solved for the case of a periodic force applied transversely by the bow at any given position. The equations of motion of the bridge are next written and dealt with. The *modus operandi* of the bow is afterwards examined and a simplified kinematical theory of the bowed string is based upon it. . . . Another interesting subject here treated is that of the effect of the *mute*, which, by loading the bridge, enfeebles and veils the tone of the instrument. . . . This work contains twenty-eight figures in the text and twenty-six excellent full-page photographic reproductions, and well deserves the careful attention of those interested in such a notable contribution to an important subject."]

**SCIENTIA,** volume 84, April, 1919: "Synthèses et visions d'histoire de la science" by A. Mieli, 310–313; [Discussion of topic with special reference to three works: S. Günther, *Geschichte der Naturwissenschaften*, 2 little volumes, Leipzig, Reclam; F. Dannemann, *Die Naturwissenschaften in ihrer Entwickelung und ihrem Zusammenhang*, 4 volumes, 1910–1913; and W. Libby, *An Introduction to the History of Science*, New York, 1918.] Review by G. Scorza of Borel's *Introduction géométrique à quelques théories physiques* (Paris, 1914) and Borel's *Leçons sur les fonctions monogènes uniformes d'une variable complexe* (Paris, 1917), 314–316; Review by R. Marcolongo of Whittaker's *A Treatise on the analytical dynamics of particles and rigid bodies* (Cambridge, 1917).

**SCIENTIFIC AMERICAN SUPPLEMENT,** volume 87, April 5, 1919: "A scientific coin problem. How some mysterious numerical effects can be produced" by T. L. De Land, 220.

**TEXAS MATHEMATICS TEACHER'S BULLETIN,** volume 4, no. 3, May, 1919: "Summer school courses in mathematics," 3–6; "Number systems and numerals" by Phillis Henry, 7–10; "Saving time and gaining efficiency in teaching trigonometry" by Goldie P. Horton, 11–14; "A few facts about mathematics" by P. M. Batchelder, 15–17; "Bibliography of the teaching of mathematics" by C. D. Rice, 18–20; "Methods *vs.* results and mathematical induction" by A. A. Bennett, 21–28; "A trigonometric solution of the irreducible case of the cubic equation with applications" by C. E. Brand, 29–34; "Checking the solution of an equation" and "Familiar tricks based on literal arithmetic," 35–38—[Reprinted from W. F. White's *A Scrap Book of Elementary Mathematics*]; "The Pentagram during 1918–1919" by Goldie P. Horton, 39–40; "The straight edge," 41.

**TÔHOKU MATHEMATICAL JOURNAL,** volume 15, nos. 1–2, March, 1919: "On a determinate system of non-independent trials" by M. Watanabe, 1–134; "Sur l'équation $x_1^5 + x_2^5 + x_3^5 + \cdots + x_8^5 = 0$" by A. Filippov, 135–145; "On closed curves described by a spherical pendulum" by A. Emch, 146–165; "On a transformation theorem relating to spheroidal harmonics" by B. Datta, 166–171; "On certain mean curves defined by the series of orthogonal functions" by K. Ogura, 172–180; "A remark on the dynamical system with two degrees of freedom" by K. Ogura, 181–183; "A contribution to the question of linear dependence in linear integral equations" by L. J. Rouse, 184–216.

**ZEITSCHRIFT FUR MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHT ALLER SCHULGATTUNGEN,** volume 49, no. 12, December, 1918: "Die Entwicklung des Satzes vom vollständigen Vierseit und Viereck zu einem Grundpfeiler des natürlichen Systems der Geometrie" by W. Dieck, 329–341; "Über die Anwendung beweglicher Figuren im geometrischen Unterricht" by Elisabeth K. Staiger, 341–348; Reviews of Neuendorff's *Praktische Mathematik*, 2. Teil (Leipzig, 1918), and of Seyffarth's *Elementarmathematik zum Gebrauch an Lehrerbildungsanstalten*, 1. Teil: Allgemeine Arithmetik und Algebra, 5. Auflage (Dresden–Blasewitz, 1917).

## AMERICAN DOCTORAL DISSERTATIONS.

GILLIE A. LAREW, *Necessary conditions in the problems of Mayer in the calculus of variations.* Pp. 1–22. [Reprinted from *Transactions of the American Mathematical Society*, 1919.] (Chicago, 1916.)

A. S. MERRILL, *An isoperimetric problem with variable end-points.* Pp. 60–78. [Reprinted from *American Journal of Mathematics*, 1919.] (Chicago, 1916.)

PAULINE SPERRY, *Properties of a certain projectively defined two-parameter family of curves on a general surface.* Pp. 213-214. [Reprinted from *American Journal of Mathematics*, 1918.] (Chicago, 1916.)

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## UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY U. G. MITCHELL, University of Kansas, Lawrence.

### CLUB ACTIVITIES.

BARNARD COLLEGE MATHEMATICS CLUB, Columbia University, New York, N. Y. [*1918*, 226–227].

The officers of the club for 1918–19 are: Honorary president, Professor Edward Kasner; president, Janet Meneely '19; vice-president, Veronica Jentz '20; secretary, Georgia Schaaf '19; treasurer, Evelyn Baldwin; program committee: Gretchen Torek '19, Catherine Piersall '20, Marian Haskell '21.

The following are programs of meetings for 1918–19:

November 19, 1918: "Theory of ballistics" by Professor Edward Kasner.

December 16: "Descartes" by Alice Johnson '21 and Marian Haskell '21; "Leibniz" by Helen Clarke '20; "Newton" by Mimosa Pfaltz '19.

January 14, 1919: "Occupations open to women taking mathematics" by Ellen Leut '18.

February 18: "Cycloids" by George W. Mullins, instructor in mathematics.

March 11: "Physics and mathematics" by Elaine Kennard '20; "Relation of curves to physics" by Catherine Piersall '29.

April 16: "Accounting" by Professor Wildman, New York University.

May: Business meeting.

DENISON MATHEMATICS CLUB, Denison University, Granville, Ohio. [*1918*, 403–04].

The officers of the club for the year 1918–19 are: President, Edgar King '19; vice-president, Esther Weaver '20; secretary-treasurer, Fern Whitney '21.

The programs for the first part of the year 1919 were as follows:

January 21, 1919: "The significance of mathematics" (excerpts from a paper by Professor Cassius J. Keyser[1]) by Professor Anna B. Peckham.

February 4: "Parallel coördinates" by Professor Forbes B. Wiley.

February 18: "Mathematics in artillery" by Lieut. R. Thompson '21.

March 4: "Fundamentals in mathematics" by Professor Charles C. Morris, Ohio State University.

March 18: "Parallel coördinates" (continued) by Professor Wiley.

April 15: "Centrodes" by Esther Weaver '20; "Magic squares" by Alva Shumaker '22.

April 29: "Applications of mathematics to astronomy" by Professor Paul Biefeld, department of astronomy, Denison University; election of officers.

[1] "The Human Significance of Mathematics," *Science*, new ser., Vol. 42, No. 1089 (Nov. 12, 1915), pp. 663–680.

May 13: First annual banquet.

The club has about ten dollars in its treasury which it plans to use in purchasing portraits of noted mathematicians for the walls of the mathematics rooms.

The officers elected for the year 1919–20 are: President, Richard Howe '20; vice-president, Esther Weaver '20; secretary, Evangeline Nellis '22; treasurer, Alva Shumaker '22.

UNDERGRADUATE MATHEMATICS CLUB, University of Illinois, Urbana, Ill. [*1918*, 404–405].

Owing to the war conditions no meetings of the club were held during 1918–19 until February 5, 1919, when the following officers were elected: President, Irene Doyle '19; vice-president, Laura Stoll '19; secretary-treasurer, Beulah Prante '19.

Membership is limited to juniors and seniors majoring in mathematics and to those students who have attended three consecutive meetings. The average attendance is about twelve. The following programs have been given since February fifth.

February 19, 1919: "The training value of mathematics" by Beulah Prante '19.
March 5: "Mathematical puzzles" by Margaret Walker '19.
March 19: "Women mathematicians" by Irene Doyle '19.
April 9: "Maxima and minima" by Agnes Nelson '19.

THE WHITE MATHEMATICS CLUB AT THE UNIVERSITY OF KENTUCKY, Lexington, Ky. [*1918*, 90, 451–345].

The following is the list of programs for the first part of the year 1919.

January 13, 1919: "Geometric determination of $\pi$" by Professor Elijah L. Rees.
January 20: "Analytic determination of $\pi$" by Professor Harold H. Downing.
January 27: "Transcendence of $e$" by Professor Harold H. Downing.
February 10: "Euclid's Elements" by H. P. Pettit Gr.
February 26: "Transcendence of $\pi$" by Guy W. Smith, instructor in mathematics.
March 3: "A non-euclidean world" by W. W. Elliott Gr.; "Discussion of problems" by H. P. Pettit Gr.
March 10: "Introduction to linear associative algebras" by Dr. Guy W. Smith; "Solution of problems" by Professor Downing and H. P. Pettit Gr.
March 17: "History of the parallel postulate"[1] by Mary E. Beall '19; "Discussion of problems" by H. P. Pettit Gr.
March 26: "Some applications of vector analysis to the theory of ruled surfaces" by Professor Rees.
March 31: "Logical significance of definitions, axioms and postulates" by Edna Berkele '19.
April 7: "The net of Moebius" by Professor Paul P. Boyd.

[1] The subjects discussed by undergraduates are selected from J. W. Young's *Fundamental Concepts of Algebra and Geometry* (Macmillan, 1911).

April 14: "Consistency, independence and categoricalness of a set of assumptions" by Frank W. Tuttle '20.

April 21: "A problem in partial differential equations" by Professor J. Morton Davis.

April 28: "Class, correspondence, number" by Frances Kimbrough '20.

## CLUB NOTES.

A number of inquiries have been received asking if the editor of this department can suggest some mathematical games for the use of clubs at social meetings. Can not some of our readers supply us with descriptions of mathematical games or plays which they have found successful? The editor gives below several forms of entertainment, more or less mathematical, which have been used in social meetings of the club of which he is a member.[1]

*Mathematical contests.* Two captains choose sides as in an old-fashioned spelling match. The captain of each side designates a contestant to represent his side and the two contestants stand with their backs to the blackboard[2] while a member of the faculty places on the board back of one contestant a mathematical exercise and an assistant copies the same exercise on the board back of the other contestant. At a given signal the contestants turn and attack the exercise. The one first obtaining the correct answer is declared the winner and must defend his position against the next contestant designated by the captain of the losing side. The losing contestant drops out of the game. The play is continued until all of the contestants on one team have been defeated and the side whose representative is still undefeated is declared the winner of the contest. By continuing the play among the undefeated members of the winning team the champion of the club can be determined.

Where membership in the club is limited to upperclassmen, there is considerable latitude in the choice of exercises, but the use of simple exercises in arithmetic and participation in the contest by all members of the faculty as well as by all of the students are likely to produce a merrier time.

*Telling fortunes by mathematics.* *The Ancient Science of Numbers*, by Luo Clement (New York, Rogers Bros., 1909) is a book which is not likely to be taken very seriously by many of our readers; but it contains a system, taken seriously by its author, for telling something of the "apparently mysterious operation of the Science of Numbers, and its effect upon the health, happiness and success of the individual" which, in the hands of a clever operator, who has

[1] The *Teachers College Record*, New York, for November, 1912 (Vol. 13, pp. 385–495), is devoted to articles on "Number Games and Number Rhymes" by D. E. Smith, C. W. Hunt, F. J. Flynn, C. C. Eaton, R. K. Atwell, and F. B. Selkin. Chapter 3 (pp. 413–422) on "Rithmomachia, the great medieval number game" by D. E. Smith and C. C. Eaton, is reprinted, with a few modifications from this MONTHLY, April, 1911; a bibliography is given on page 495.

References to the literature of "Nim, a Game with a Complete Mathematical Theory" were given, in this MONTHLY, in Topics for Club Programs—No. 7, March, 1918. "Probabilities in the game of 'Shooting Craps'" will be discussed in the next number of the MONTHLY.—Editor-in-Chief.

[2] The meetings attended by the editor were held in a private home and a portable blackboard was brought in for the evening.

imagination and the ability to act a part gravely, may be made the basis of a pleasant evening's entertainment. The writer has seen it used on more than one occasion, but most successfully as the main feature of a Halloween party with a member of the faculty taking the part of the fortune-teller. The house was decorated with jack-o'-lanterns, paper witches, black cats, etc., and an appropriate booth in a corner of a room was provided for the fortune-teller. The "fortunes" told were based on the "Ancient Science of Numbers" but obviously supplemented with much other real and imaginary information more or less surreptitiously obtained by the amateur revealer of mysteries.

*An evening at chess.* Chess is essentially a mathematical game, as is evidenced by the fact that the Royal Society Index lists no less than thirty mathematical papers on the "Knight's Move" alone. Mathematical students generally are fond of the game or enjoy learning it when not already players.

For an evening at chess the first requisite is a sufficient number of boards and sets of men so that every member can play. Before the play is begun a member of the club presents a brief sketch of the history of the game and the mathematical problems connected with it. The club is then divided into players and beginners. If convenient, the players may go into one room and the beginners into another. The players are seated at small tables and the play proceeds as in a progressive card game except that the time for play is limited (say to 10 or 15 minutes) and one of the best players, who has been chosen to act as judge and timekeeper, awards all unfinished games to the player who, in his judgment, holds the more advantageous position.

The beginners include all who have not yet learned the game. Two or three experienced players explain the game to them, illustrating by means of simple games based on easy checkmates such as the "fool's mate" and the "scholar's mate." After sufficient instruction has been given, the beginners divide into pairs and spend the evening playing, their instructors remaining in attendance to answer questions or give other assistance.

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## PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

**Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.**

### PROBLEMS FOR SOLUTION.

**2780. Proposed by ELMER LATSHAW, West Philadelphia, Pa.**

A polygon whose sides are $a$, $2a$, $3a$, $4a$ is inscribed in a circle. Find the radius of the circle.

**2781. Proposed by J. L. RILEY, Stephensville, Texas.**

Show that the asymptotic lines on a pseudospherical surface are curves of constant torsion.

**2782. Proposed by WARREN WEAVER, University of Wisconsin.**

A great number $N$ of jackstraws are jumbled up in such a way that any one is as likely to have one direction as another. Show that the probable number that make an angle lying between $\theta_1$ and $\theta_2$ as measured from any given direction is equal to $\frac{N(\cos\theta_1 - \cos\theta_2)}{2}$.

**2783. Proposed by C. C. BRAMBLE, U. S. Naval Academy.**

Two players $A$ and $B$ take turns throwing a single die, $A$ leading. The one first making a score of 3 aces is to be the winner. Find the probability that $A$ will win.

**416 (Algebra) [May, 1914]. Proposed by C. E. FLANAGAN, Wheeling, Va.**

The sides of a given rectangle are $a$ and $b$ in which a rectangle is to be inscribed one of whose sides is $c$. Find the other side, using Euler's rule for quartics.

**353 (Calculus) [February, 1914]. Proposed by R. P. LOCHNER, Philadelphia, Pa.**

The center of a sphere, radius $R = 5$ inches, is $a = 10$ inches above the surface of a sphere, radius $12\frac{1}{2}$ inches. There is a point of light at $b = 1$ inch horizontally from a point $c = 10$ inches vertically above the surface of the first sphere. What is the area of the shadow which the upper sphere casts on the lower one?

**287 (Mechanics) [February, 1914]. Proposed by W. H. DRANE, Lebanon, Tenn.**

While sitting in an empaled enclosure, I noticed that the spokes of the wheels of passing automobiles, when viewed through the pickets of the fence, appeared to revolve more slowly than they really did, and in some instances even appeared to be revolving in a direction opposite to that in which they were really turning. Explain this optical illusion.

**202 (Number Theory) [December, 1913]. Proposed by A. R. SCHWEITZER, Chicago, Ill.**

There exists an infinitude of systems of dyads $\{\alpha\beta\}$ in 7, 9, 11, etc., elements such that each system has the following properties: (1) if $\alpha\beta$ is in the set, then $\beta\alpha$ is not in the set; (2) for each dyad $\alpha\beta$ in the set there exists an element $\xi$ such that $\xi\beta$ and $\alpha\xi$ are also in the set. For example, such a system is,

12, 23, 34, 45, 56, 67, 78, 89, 91
13, 24, 35, 46, 57, 68, 79, 81, 92
14, 25, 36, 47, 58, 69, 71, 82, 93
51, 62, 73, 84, 95, 16, 27, 38, 49.

Investigate the existence of

I. A finite set of triads $\{\alpha\beta\gamma\}$ such that (1) if $\alpha\beta\gamma$ is in the set, then $\beta\gamma\alpha$, $\gamma\alpha\beta$ are also in the set but $\beta\alpha\gamma$ is not in the set; (2) for each triad $\alpha\beta\gamma$ in the set there exists an element $\xi$ such that $\xi\beta\gamma$, $\alpha\xi\gamma$, $\alpha\beta\xi$ are also in the set.

II. A finite set of tetrads $\{\alpha\beta\gamma\delta\}$ such that (1) if $\alpha\beta\gamma\delta$ is in the set, then $\beta\gamma\alpha\delta$, $\gamma\alpha\beta\delta$, $\gamma\delta\alpha\beta$ are also in the set but $\beta\alpha\gamma\delta$ is not in the set; (2) for each tetrad $\alpha\beta\gamma\delta$ in the set there exists an element $\xi$ such that $\xi\beta\gamma\delta$, $\alpha\xi\gamma\delta$, $\alpha\beta\xi\delta$, $\alpha\beta\gamma\xi$ are also in the set.

The problem for alternating $n$-ads for $n > 4$ is obvious.

## SOLUTIONS OF PROBLEMS.

**2699, 2710 [May, June, 1918; April, 1919]. Proposed by the late R. E. MOORE, University of Wisconsin.**

If $a_k^{(r)}$ denotes the $k$th term of an arithmetic progression of order $r$, and $c_k$ denotes the $k$th binomial coefficient in the expansion of $(a - b)^n$ ($n$ being a positive integer), show that

$$s \equiv \sum_{k=1}^{n+1} c_k a_k^{(r)} = 0, \text{ if } n > r.$$

### II. Solution by A. Pelletier, Montreal, Canada.

The terms of the arithmetic progression may be expressed in the usual form as follows:

$$a, \quad a + \Delta_1, \quad a + 2\Delta_1 + \Delta_2, \quad a + 3\Delta_1 + 3\Delta_2 + \Delta_3, \quad \cdots.$$

We have to prove that

$$c_0 a + c_1(a + \Delta_1) + c_2(a + 2\Delta_1 + \Delta_2) + c_3(a + 3\Delta_1 + 3\Delta_2 + \Delta_3)$$
$$+ \cdots + c_n\left(a + n\Delta_1 + \frac{n(n-1)}{1\cdot 2}\Delta_2 + \cdots\right) = 0,$$

or that

$$\begin{aligned}
&a[c_0 + c_1 + c_2 + c_3 + \cdots + c_n] \\
&+ \Delta_1[c_1 + c_2 \cdot 2 + c_3 \cdot 3 + \cdots + c_n \cdot n] \\
&+ \Delta_2\left[c_2 + c_3 \cdot 3 + c_4 \cdot 6 + \cdots + c_n \frac{n(n-1)}{2!}\right] \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&+ \Delta_k[c_k + c_{k+1} \cdot c_1^{k+1} + c_{k+2} \cdot c_2^{k+2} + \cdots + c_n \cdot c_{n-k}^{n}] = 0,
\end{aligned}$$

where $k$ is used for $r$, and $c_{n-k}^{n} = \binom{n}{n-k}$. Now, each of the quantities in the brackets is equal to zero; for, the general form of the bracketed expressions is $c_k + c_{k+1} \cdot c_1^{k+1} + c_{k+2} \cdot c_2^{k+2} + \cdots + c_n^{n} \cdot c_{n-k}$ and this is equal to $c_k[1 - c_1^{n-k} + c_2^{n-k} - \cdots (-1)^{n-k} c_{n-k}^{n-k}] = c_k^{n}(1-1)^{n-k} = 0$, for $n > k$.

Also solved by A. M. HARDING, E. E. WITMER, and the Proposer.

*Note by the Editor:* The solution of Mr. Pelletier does not differ greatly from solution I by Mr. E. H. Clarke. Problem 481 (Algebra) [May, 1917], also proposed by Mr. Moore, is really the same as problem 2699 (or 2710), and has been solved by Professor F. H. SAFFORD.

**2702 [May, 1918]. Proposed by N. P. PANDYA, Sojitra, India.**

A conic of variable eccentricity has a focus and corresponding directrix fixed. The latus rectum cuts a fixed circle in $A'$ and $B'$. If $A$ be the vertex of the conic, find the locus of the centroid of the triangle $AA'B'$.

SOLUTION BY H. L. OLSON, Chicago, Illinois.

If the directrix be taken as the $y$ axis, and the perpendicular from the focus to the directrix as the $x$ axis, the equation of the conic may be written,

$$\{(1-e^2)x - k\}^2 + (1-e^2)y^2 = k^2e^2,$$

where $k$ is the (fixed) perpendicular distance from the focus to the directrix. Let the equation of the fixed circle be $(x-p)^2 + (y-q)^2 = r^2$. The latus rectum, $x = k$, of the conic cuts this circle in the points for which $x = k$, $y = q \pm \sqrt{r^2 - (k-p)^2}$. The coördinates of the vertex $A$ of the conic are $x = k/(1+e)$, $y = 0$. Hence, the coördinates of the centroid of the triangle $AA'B'$ are

$$x = \tfrac{2}{3}k + \frac{k}{3(1+e)}, \qquad y = \tfrac{2}{3}q,$$

and its locus is the straight line $y = \tfrac{2}{3}q$.

Also solved by A. M. HARDING.

**2703 [May, 1918]. Proposed by S. A. COREY, Albia, Iowa.**

Let $A_1$, $A_2$, $A_3$, $A_4$ and $-(A_1 + A_2 + A_3 + A_4)$ be the vector sides of a pentagon, plane or gauche. Let $B_1$, $B_2$, $B_3$, $B_4$ and $-(B_1 + B_2 + B_3 + B_4)$ be the sides of a second pentagon where

$$B_1 = (c_1 + c_2)A_1 + (c_1 - c_2)A_3, \qquad B_2 = (c_4 + c_3)A_4 + (c_4 - c_3)A_2,$$
$$B_3 = (c_2 + c_1)A_2 + (c_2 - c_1)A_4, \qquad B_4 = (c_3 + c_4)A_3 + (c_3 - c_4)A_1,$$

$c_1$, $c_2$, $c_3$, and $c_4$ being ordinary scalars.

Then, if $a_r$ and $b_s$ be the lengths of the sides $A_r$ and $B_s$ respectively and if $\cos A_rA_s$ denotes the cosine of the angle included between the sides $A_r$ and $A_s$ and if $\cos B_rB_s$ denotes the cosine of the angle included between the sides $B_r$ and $B_s$ prove that

$$2(c_1c_4 + c_2c_3)(a_1a_4 \cos A_1A_4 + a_2a_3 \cos A_2A_3) = b_1b_2 \cos B_1B_2 + b_3b_4 \cos B_3B_4.$$

SOLUTION BY LOUIS WEISNER, Student, College of the City of New York.

From the given equations, we have

$$B_1 \cdot B_2 = (c_1c_4 + c_1c_3 + c_2c_4 + c_2c_3)A_1 \cdot A_4 + (c_1c_4 - c_1c_3 + c_2c_4 - c_2c_3)A_1 \cdot A_2 + (c_1c_4 + c_1c_3 - c_2c_4 - c_2c_3)A_3 \cdot A_4 + (c_1c_4 - c_1c_3 - c_2c_4 + c_2c_3)A_3 \cdot A_2$$

$$B_3 \cdot B_4 = (c_2c_3 + c_2c_4 + c_1c_3 + c_1c_4)A_2 \cdot A_3 + (c_2c_3 - c_2c_4 + c_1c_3 - c_1c_4)A_2 \cdot A_1 + (c_2c_3 + c_2c_4 - c_1c_3 - c_1c_4)A_4 \cdot A_3 + (c_2c_3 - c_2c_4 - c_1c_3 + c_1c_4)A_4 \cdot A_1.$$

Adding,

$$B_1 \cdot B_2 + B_3 \cdot B_4 = (2c_1c_4 + 2c_2c_3)A_1 \cdot A_4 + (2c_1c_4 + 2c_2c_3)A_2 \cdot A_3 = 2(c_1c_4 + c_2c_3)(A_1 \cdot A_4 + A_2 \cdot A_3).$$

By definition $M \cdot N = mn \cos MN$, where $m$ and $n$ are the lengths of $M$ and $N$, respectively, and $\cos MN$ is the cosine of the angle between the vectors $M$ and $N$.

Hence,

$$b_1b_2 \cos B_1B_2 + b_3b_4 \cos B_3B_4 = 2(c_1c_4 + c_2c_3)(a_1a_4 \cos A_1A_4 + a_2a_3 \cos A_2A_3).$$

Also solved by the PROPOSER.

**2704 [May, 1918]. Proposed by WILLIAM HOOVER, Columbus, Ohio.**

A particle moves from rest under gravity down the arc of a parabola with the axis vertical and concavity upward; express the time to the vertex in terms of an elliptic integral of the second kind.

SOLUTION BY THE PROPOSER.

Taking the $y$-axis along the axis of the curve, the equation of the parabola is

$$x^2 = 4ay. \tag{1}$$

In the usual notation, the equation of motion of the particle is

$$mv\frac{dv}{ds} = -mg\frac{dy}{ds}.$$

Multiplying by $2ds$ and integrating,

$$v^2 = -2gy + C. \tag{2}$$

When $v = \dot{s} = 0$, let $y = h$; then $C = 2gh$, and (2) is

$$\dot{s}^2 = 2g(h - y). \tag{3}$$

From (1),

$$\dot{s} = \sqrt{\frac{a + y}{y}}\,y$$

and (3) becomes

$$dt = -\frac{1}{\sqrt{2g}}\sqrt{\frac{a + y}{y(h - y)}}\,dy. \tag{4}$$

Now change the variable by setting $y = h - hy$; then

$$dt = \frac{1}{\sqrt{2g}}\sqrt{\frac{(a + h) - hy}{y(1 - y)}}\,dy. \tag{5}$$

The upper and lower limits in (4) are $y = h$, $y = 0$, and so (5) gives

$$t = \frac{\sqrt{h}}{\sqrt{2g}}\int_0^1 \sqrt{\frac{\frac{a + h}{h} - y}{y - y^2}}\,dy = \sqrt{\frac{2(a + h)}{g}}\,E\left(\sqrt{\frac{h}{a + h}}\right).$$

Also solved by R. C. COLWELL and A. M. HARDING.

**2705 [May, 1918]. Proposed by PAUL CAPRON, U. S. Naval Academy.**

Find the area of a loop of the trochoid

$$x = \tfrac{1}{3}a(3\phi - \pi \sin \phi), \qquad y = \tfrac{1}{3}a(3 - \pi \cos \phi).$$

SOLUTION BY L. E. MENSENKAMP, Freeport, Illinois.

The curve of the type here given is generated by causing a circle of radius $a$ to roll on a straight line. Then, a point on a fixed radius of this circle, and at a distance $\pi a/3$ from the center, will describe the given trochoid. The axis of $X$ is the line on which the circle rolls, and the axis of $Y$ is taken as the position of the radius containing the generating point when it is pointing vertically downward.

The desired area may be found by evaluating $\int y\,dx$ between the proper limits of integration. It is convenient to express this integral in terms of the parameter $\phi$, as

$$\int \frac{a^2}{9}(3 - \pi \cos \phi)^2 d\phi.$$

From the way in which the curve is defined above, it is symmetrical with respect to the $Y$-axis; therefore, it will suffice to find that portion of the loop to the left of the $Y$-axis and multiply it by 2. As $\phi$ starts from zero and increases in value, the point which generates the curve starts from its position on the negative portion of the $Y$-axis and traverses the left half of the loop in a clock-wise direction about the origin, finally returning to the $Y$-axis again, this time in the upper half of the coördinate plane. The limits of integration in $\phi$ must, therefore, correspond to the two points where the loop cuts the $Y$-axis. Examination of the equations of the curve shows these to be 0 and $\pi/6$.

The area of half the curve is therefore given by

$$\frac{a^2}{9}\int_0^{\pi/6}(9 - 6\pi \cos \phi + \pi^2 \cos^2 \phi)d\phi,$$

or

$$\frac{a^2}{9}\left(9\phi - 6\pi \sin \phi + \frac{\pi^2\phi}{2} + \frac{\pi^2}{4}\sin 2\phi\right)_0^{\pi/6}.$$

The area of the entire loop will then be double this, or

$$\frac{a^2\pi}{108}(-36 + 2\pi^2 + 3\pi\sqrt{3}).$$

Also solved by J. B. REYNOLDS and the PROPOSER.

**2707 [May, 1918]. Proposed by S. A. COREY, Albia, Iowa.**

Let $A$, $B$, and $C$ be the vector sides of a triangle. Construct another triangle with vector sides $R$, $S$, and $T$, where

$$R = mA - dnB, \qquad S = nA + (m + en)B.$$

Then prove that

$$(m^2 + emn + dn^2)(a^2 + eab\cos(AB) + db^2) = r^2 + ers\cos(RS) + ds^2,$$

where $d$, $e$, $m$ and $n$ are any scalar quantities; $a$, $b$, $r$ and $s$ are the tensors, or lengths, of the sides $A$, $B$, $R$ and $S$, respectively; and $\cos(RS)$ is the cosine of the angle between $R$ and $S$ when placed coinitially.

SOLUTION BY THE PROPOSER.

We have the algebraic identity

$$(m^2 + emn + dn^2)(a^2 + eab + db^2) = r^2 + ers + ds^2, \tag{1}$$

where $r = ma - dnb$, and $s = na + (m + en)b$.

As (1) is a quadratic identity in $a$, $b$, $r$, and $s$, it holds if these quantities be vectors (or quaternions), provided we consider the scalar part only of the vector products which we would have in (1) under such an interpretation. But such an interpretation gives us exactly the equation of the given problem, which is therefore true.

NOTE.—I have used capitals $A$, $B$, $C$, $R$, $S$, and $T$ in some cases where small letters are used in the problem as published in the May MONTHLY. This substitution of capitals for small letters is done for the sake of clearness in distinguishing vectors from their tensors.

**2708 [May, 1918]. Proposed by WILLIAM HOOVER, Columbus, Ohio.**

A uniform plank of length $2a$ and thickness $2h$ rests in equilibrium on a fixed rough horizontal cylinder of radius $c$, so that a vertical plane containing the dimension $2a$ and the center of gravity of the plank is at right angles to the axis of the cylinder; find the period of a complete small oscillation of the plank.

## SOLUTION BY THE PROPOSER.

Let $G$ be the center of gravity of the plank; $G_0$ the initial position of $G$; $O$ the center of gravity of the cylinder and vertically beneath $G_0$; $\varphi$ = the angular rotation of the plank after any time $t$ from the beginning of motion, $OX$, $OY$ the horizontal and vertical coördinate axes $OA = x$, $GA = y$, the coördinates of $G$, $k = \sqrt{(a^2 + h^2)/3}$ = the radius of gyration of the plank about an axis parallel to the axis of the cylinder. We obtain

$$x = (c + h) \sin \varphi - c\varphi \cdot \cos \varphi, \tag{1}$$

$$y = (c + h) \cos \varphi + c\varphi \cdot \sin \varphi. \tag{2}$$

By *vis viva*,

$$\tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + k^2\dot{\varphi}^2) = -mgy + C. \tag{3}$$

From (1),

$$\dot{x} = (h \cos \varphi + c\varphi \sin \varphi)\dot{\varphi}, \tag{4}$$

$$\dot{y} = (-h \sin \varphi + c\varphi \cos \varphi)\dot{\varphi}; \tag{5}$$

(3) then is

$$\frac{m}{2}(h^2 + k^2 + c^2\varphi^2)\dot{\varphi}^2 = C - mg\{(c + h) \cos \varphi + c\varphi \sin \varphi\}. \tag{6}$$

Differentiating both sides of (6) with respect to $t$ using the value of $k$,

$$3c^2\varphi \cdot \dot{\varphi}^2 + \{(a^2 + 4h^2) + c^2\varphi^2\}\ddot{\varphi} = -3g\{-h \sin \varphi + c\varphi \cos \varphi\}. \tag{7}$$

Let $\varphi$ be so small that we may put $\sin \varphi = \varphi$, $\cos \varphi = 1$, and omit $\varphi^2$, $\dot{\varphi}^2$, etc., because of the nature of the oscillation; we have, then,

$$(a^2 + 4h^2)\ddot{\varphi} = -3g(c - h) \cdot \varphi, \tag{8}$$

an harmonic equation in $\varphi$ if $c > h$, giving the period required,

$$T = 2\pi\sqrt{\frac{a^2 + 4h^2}{3g(c - h)}}. \tag{9}$$

It may be instructive to derive (8) by another method given by Holditch in the eighth volume of the *Cambridge Transactions* and quoted by Routh, *Dynamics of a System of Rigid Bodies, Elementary Part*, fourth edition, 1882, pages 341–342.

Let the motion of a body in space of two dimensions be given by the coördinates $x$, $y$ of its center of gravity, and the angle $\varphi$ which any fixed line in the body makes with a line fixed in space; $\alpha$ = the equilibrium value of $\varphi$; $x'$, $x''$, etc., denoting $dx/d\varphi$, etc.; $x_0'$, etc., the values of $x'$, etc., when $\varphi = \alpha$, and $k$ = the principal radius of gyration; then

$$(x_0'^2 + k^2)\ddot{\varphi} = -gy_0''\varphi. \tag{10}$$

From (4) and (5) we have, with $\varphi_0 = 0$,

$$x_0' = h, \qquad y_0'' = c - h \tag{11}$$

(11) in (10) gives (8).

Also solved by R. C. COLWELL and J. B. REYNOLDS.

**2711 [June, 1918]. Proposed by PAUL CAPRON, U. S. Naval Academy.**

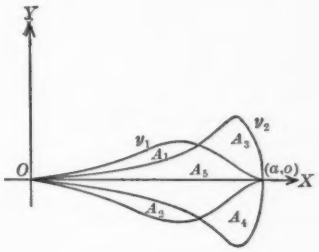
Show that the curves (*a*) $a^2y_1^2 = x^4(a^2 - x^2)^3$, (*b*) $a^3y_2^2 = x^3(a^2 - x^2)$ bound ten areas, of which two are each $(a^2/4)(\frac{1}{4}\pi - \frac{1}{3})$ and the remaining eight are each $a^2/24$.

SOLUTION BY A. M. HARDING, University of Arkansas.

Since each of the curves is symmetrical with respect to both axes we shall consider only those areas which lie on the positive side of the $x$-axis.

$$A_1 = A_2 = \int_0^{a\sqrt{1/2}} (y_1 - y_2)dx = \frac{1}{a^4}\int_0^{a\sqrt{1/2}} [x^2(a^2 - x^2)^{3/2} - x^4(a^2 - x^2)^{1/2}]dx$$

$$= \frac{1}{a^4}\int_0^{a\sqrt{1/2}} x^2\sqrt{a^2 - x^2}(a^2 - 2x^2)dx.$$




Let

$$x = a\sin\frac{\theta}{2}.$$

Then

$$A_1 = A_2 = \frac{a^2}{8}\int_0^{\pi/2} \sin^2\theta\cos\theta\, d\theta = \frac{a^2}{24}.$$

$$A_3 = A_4 = \int_{a\sqrt{1/2}}^{a} (y_2 - y_1)dx = \frac{1}{a^4}\int_{a\sqrt{1/2}}^{a} x^2\sqrt{a^2 - x^2}(2x^2 - a^2)dx.$$

Let $x^2 = a^2 - u^2$. Then

$$A_3 = A_4 = \frac{1}{a^4}\int_0^{a\sqrt{1/2}} u^2\sqrt{a^2 - u^2}(a^2 - 2u^2)du = \frac{a^2}{24},$$

as above.

$$\tfrac{1}{2}A_5 = \int_0^{a\sqrt{1/2}} y_2dx + \int_{a\sqrt{1/2}}^{a} y_1dx = \frac{1}{a^4}\int_0^{a\sqrt{1/2}} x^4\sqrt{a^2 - x^2}dx + \frac{1}{a^4}\int_{a\sqrt{1/2}}^{a} x^2(a^2 - x^2)^{3/2}dx.$$

The last integral reduces to

$$\frac{1}{a^4}\int_0^{a\sqrt{1/2}} u^4\sqrt{a^2 - u^2}du$$

on setting $x^2 = a^2 - u^2$. Hence,

$$\tfrac{1}{2}A_5 = \frac{2}{a^4}\int_0^{a\sqrt{1/2}} x^4\sqrt{a^2 - x^2}dx = \frac{a^2}{8}\left(\frac{\pi}{4} - \frac{1}{3}\right),$$

or

$$A_5 = \frac{a^2}{4}\left(\frac{\pi}{4} - \frac{1}{3}\right).$$

Also solved by A. R. NAUER, H. L. OLSON, and the Proposer.

**2712 [June, 1918]. Proposed by WILLIAM HOOVER, Columbus, Ohio.**

Given the conic $ax^2 + 2hxy + by^2 - 2x = 0$. Find the locus on which lie the four points of intersection of pairs of tangents to the conic from a pair of points on the $x$-axis equidistant from the origin.

SOLUTION BY A. H. WILSON, Haverford College.

The pair of tangents to the conic $C \equiv ax^2 + 2hxy + by^2 - 2x = 0$ from the point $(\alpha, 0)$ are represented by the equation $\lambda C + l^2 = 0$, where, $l \equiv (a\alpha - 1)x + h\alpha y - \alpha = 0$, is the polar of $(\alpha, 0)$, and $\lambda = \alpha(2 - a\alpha)$. Similarly, if $m \equiv (a\alpha + 1)x + h\alpha y - \alpha = 0$, $\mu = -\alpha(2 + a\alpha)$, $\mu C + m^2 = 0$, represents the tangents from $(-\alpha, 0)$.

The elimination of $\alpha$ from the equations $\lambda C + l^2 = 0$ and $\mu C + m^2 = 0$ is effected at once by subtraction and gives for the required locus the conic $hxy + by^2 - x = 0$.

As $a$ does not occur in this equation, it is the locus described for any one of the conics of the one-parameter family obtained by holding $h$ and $b$ fixed and allowing $a$ to vary.

Also solved by A. M. HARDING and the Proposer.

**2714 [June, 1918]. Proposed by H. R. HOWARD, University of St. Francis Xavier's College, Nova Scotia.**

A shuffled pack of $2(p+q)$ cards contains $2p$ honors. Show that the chance of securing exactly half the honors in taking half the pack is $[F(p, q)]^2 \div F(2p, 2q)$, where $F(p, q)$ denotes the number of different sets of $p$ cards which can be selected from $(p+q)$ cards.

Show also that if one honor is removed from the pack, the chance is not thereby affected. Is this true for the chance of getting any other assigned number of honors?

## SOLUTION BY THE PROPOSER.

We have to find first the number of ways in which we can get exactly $p$ honors in $(p+q)$ cards.

We can obviously choose our "$p$" honors in $(2p)!/(p!\,p!)$ ways and the $q$ other cards in $(2q)!/(q!\,q!)$ ways.

Hence, we can effect the required division in $[(2p)!\,(2q)!]/[(p!)^2\,(q!)^2]$ ways. Now the number of ways of taking $(p+q)$ cards from the full pack is $[2(p+q)]!/(p+q)!^2$. Thus the chance is

$$\frac{(2p)!\,(2q)!}{(p!)^2(q!)^2} \div \frac{[2(p+q)]!}{(p+q)!^2}, \tag{1}$$

*i.e.*,

$$\frac{(p+q)!^2}{(p!)^2\,(q!)^2} \div \frac{(2p+q)!}{(2p)!\,(2q)!},$$

*i.e.*,

$$[F(p, q)]^2 \div F(2p, 2q).$$

Now suppose one honor removed from the pack. Then the number of ways of taking $(p+q)$ cards from the remainder and obtaining exactly $p$ honors is

$$\frac{(2p-1)!}{(p-1)!\,(p)!} \cdot \frac{(2q)!}{(q!)^2},$$

and the chance is

$$\frac{(2p-1)!}{(p-1)!\,(p)!}\,\frac{(2q)!}{(q!)^2} \div \frac{(2p+2q-1)!}{(p+q)!\,(p+q-1)!}. \tag{2}$$

Dividing (1) by (2) we obtain unity for the quotient and this proves their equality. Let $x$ be the assigned number of honors. We shall show that the condition that the chances be equal can only be satisfied by $x = p$. With the full pack the chance will now be

$$\frac{(2p)!}{x!\,(2p-x)!} \cdot \frac{(2q)!}{(p+q-x)!\,(q-p+x)!} \div \frac{(2p+2q)!}{(p+q)!^2}. \tag{3}$$

With one honor removed the chance will be

$$\frac{(2p-1)!}{x!\,(2p-x-1)!}\,\frac{(2q)!}{(p+q-x)!\,(q-p+x)!} \div \frac{(2p+2q-1)!}{(p+q)!\,(p+q-1)!}. \tag{4}$$

Dividing (3) by (4) we see that these can only be equal if

$$\frac{p}{2p-x} = 1$$

or $x = p$, which proves our statement.

Also solved by H. L. OLSON, A. PELLETIER, and E. E. WITMER.

**2715 [June, 1918]. Proposed by H. R. KINGSTON, University of Manitoba.**

$A'$, $B'$, $C'$ are points on the sides $BC$, $CA$, $AB$, respectively, of the triangle $ABC$, and $AA'$, $BB'$, $CC'$ are concurrent in $O$. $X$, $Y$, $Z$ are the three collinear points in which, by Desargues' theorem, the corresponding sides of the triangles $ABC$ and $A'B'C'$ intersect. If $A''$, $B''$, $C''$ are the vertices of the triangle formed by the lines $AX$, $BY$, $CZ$, show that $AA''$, $BB''$, $CC''$ are concurrent.

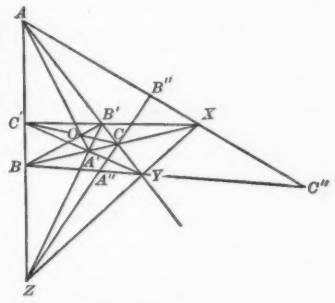
## I. Solution by William Hoover, Columbus, Ohio.

Take $ABC$ as the triangle of reference, and the trilinear coördinates of the vertices $A$, $B$, $C$ to be $(\alpha_2, 0, 0)$; $(0, \beta_2, 0)$; $(0, 0, \gamma_2)$, and those of $O$, $(\alpha_1, \beta_1, \gamma_1)$; then the equation of $AO$ is $\gamma_1\beta - \beta_1\gamma = 0$.

The coördinates of $A'$ will be proportional to $(0, \beta_1, \gamma_1)$, and, by symmetry, those of $B'$ to $(\alpha_1, 0, \gamma_1)$, and of $C'$ to $(\alpha_1, \beta_1, 0)$.

The equation of $A'B'$ is $\beta_1\gamma_1\alpha + \alpha_1\gamma_1\beta - \alpha_1\beta_1\gamma = 0$.

The point of intersection of $AB$ and $A'B'$, or $Z$, has coördinates proportional to $(-\alpha_1, \beta_1, 0)$; those of the intersection $BC$ and $B'C'$, or $X$, to $(0, -\beta_1, \gamma_1)$; and of $CA$ and $C'A'$, or $Y$, to $(\alpha_1, 0, -\gamma_1)$; and $X$, $Y$, $Z$ are collinear, since




$$\begin{vmatrix} 0, & -\beta_1, & \gamma_1 \\ \alpha_1, & 0, & -\gamma_1 \\ -\alpha_1, & \beta_1, & 0 \end{vmatrix} = \alpha_1\beta_1\gamma_1 \begin{vmatrix} 0, & -1, & 1 \\ 1, & 0, & -1 \\ -1, & 1, & 0 \end{vmatrix}$$

$$= \alpha_1\beta_1\gamma_1 \begin{vmatrix} 0, & 0, & 1 \\ 1, & -1, & -1 \\ -1, & 1, & 0 \end{vmatrix} = \alpha_1\beta_1\gamma_1 \begin{vmatrix} 1, & -1 \\ -1, & 1 \end{vmatrix} = 0.$$

The equation of $AX$ is $\gamma_1\beta + \beta_1\gamma = 0$, and those of $BY$ and $CZ$ respectively $\gamma_1\alpha + \alpha_1\gamma = 0$, and $\beta_1\alpha + \alpha_1\beta = 0$.

The coördinates of $C''$, the point of intersection of $AX$ and $BY$, are proportional to $(\alpha_1, \beta_1, \gamma_1)$; of $BY$ and $CZ$, to $(\alpha_1, \beta_1, \gamma_1)$; and of $CZ$ and $AX$ to $(\alpha_1, \beta_1, \gamma_1)$, the latter two points being $A''$, $B''$ respectively.

It is evident now that the equations to $AA''$, $BB''$, $CC''$ are the same as those of $AO$, $BO$, $CO$, in order, the first three lines then passing through $O$.

## II. Solution by H. L. Olson, Chicago, Illinois.

I shall amplify this theorem by proving that the lines $AA''$, $BB''$, $CC''$ are identical with the lines $AA'$, $BB'$, $CC'$ respectively, and hence intersect in the point $O$. In the triangles $BC'Y$ and $CB'Z$, the lines $BC$, $C'B'$, and $YZ$, joining corresponding vertices, meet at $X$, and hence the points $A$, $A'$, $A''$, in which corresponding sides meet, are collinear. Similarly $B$, $B'$, $B''$ are collinear; also $C$, $C'$, $C''$. Hence the lines $AA''$, $BB''$, $CC''$ meet at $O$.

Also solved by A. Pelletier and the Proposer.

**2716 [June, 1918]. Proposed by Clifford N. Mills, Brookings, South Dakota.**

To a passenger in a train moving at the rate of 40 miles per hour, the rain appears to be rushing downward and towards him at an angle of 20 degrees with the horizontal. If the rain is actually falling in a vertical direction, show that the velocity of the raindrops in feet per second is 21.35.

### Solution by E. H. Worthington, Elkins Park, Pa.

A velocity of 40 miles per hour is the same as $58\frac{2}{3}$ feet per second. If $v$ is the velocity of the raindrop, we have $v = 58\frac{2}{3} \tan 20° = 58\frac{2}{3} \times 0.364$ feet per sec. $= 21.35$ feet per second.

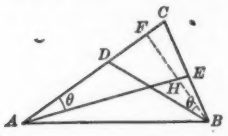
Also solved by H. E. Carleton, A. M. Harding, H. L. Olson, A. Pelletier and J. B. Reynolds.

**2735 [December, 1918]. Proposed by H. B. Phillips, Massachusetts Institute of Technology.**

If two lines $AE$ and $BD$, drawn from the vertices $A$ and $B$ of a triangle to the opposite sides, divide the angles $A$ and $B$ so that the parts of $A$ are respectively less than the corresponding parts of $B$, then $AE$ is greater than $BD$.

## SOLUTION BY THE PROPOSER.

Let $C$ be the third vertex. By hypothesis $CAE < CBD$ and $BAE < ABD$. A point $F$ can then be found on $CD$ such that $DBF = CAE$. Let $BF$ cut $AE$ in $H$. $AHF$ and $BDF$ are similar, having equal angles at $A$ and $B$ and the same angle at $F$. Therefore,




$$AH : BD = AF : BF. \tag{1}$$

Also, since $ABD > BAE$, $ABF > BAF$. Consequently,

$$AF > BF. \tag{2}$$

From (1) and (2), it follows that $AH > BD$. Therefore, $AE > AH > BD$, which was to be proved.

COROLLARY: *If $AE$ and $BD$ are equal and divide their angles in the same ratio, the triangle is isosceles.*

For, if the angles $A$ and $B$ were not equal the parts of one would be respectively less than the corresponding parts of the other and $AE$ and $BD$ would be unequal, which is contrary to hypothesis.

*In particular, if the bisectors of two angles of a triangle are equal, the triangle is isosceles.*

**2736 [December, 1918]. Proposed by M. COHEN, Freshman, Johns Hopkins University.**

Prove by elementary geometry that the orthocenter, the centroid, and the circumcenter of a triangle lie on a line (the Euler line), and that the centroid lies between the other two and is twice as far from the orthocenter as from the circumcenter.

## SOLUTION BY J. L. RILEY, Stephenville, Texas.

Let $ABC$ be the triangle under consideration; $O$ and $G$ the circumcenter and centroid, $BE$ and $CF$ perpendicular, respectively, to $AC$ and $AB$. Let mid-point of $AC$ be $B'$.

Produce $OG$ to meet the altitude $BE$ at $K$. The triangles $OGB'$ and $KGB$ are similar, for $OB'$ is parallel to $BK$, since each is perpendicular to $AC$. Then $OG : GK = B'G : GB = 1 : 2$ and hence, $GK = 2\,OG$.

If $OG$ is produced to meet the altitude $CF$ at $K'$, it follows in the same way that $GK' = 2\,OG$. Therefore, $GK' = GK$ and $K'$ coincides with $K$. Hence $BE$ and $CF$ meet at $K$ and $K$ is the orthocenter. Hence, circumcenter, centroid, and orthocenter lie on the same line.

Also solved by H. L. OLSON, C. P. SOUSLEY, and the Proposer.

**2737 [January, 1919]. Proposed by C. N. SCHMALL, New York City.**

Employing Maclaurin's theorem, or otherwise, expand the following three functions (1) $e^{\tan^{-1}x}$ as far as $x^6$; (2) $e^{\sin x}$ as far as $x^3$; and (3) $\tan x$ as far as $x^9$.

## SOLUTION BY ELMER LATSHAW, West Philadelphia, Pennsylvania.

The successive differentiation required by Maclaurin's theorem in the development of the given functions is long and laborious, but the required developments may be obtained by comparing the derivative of the function with the function itself.

Assume

$$e^{\tan^{-1}x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_6x^6 + \cdots. \tag{1}$$

Differentiating both sides of (1),

$$e^{\tan^{-1}x}\frac{1}{1+x^2} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + 6a_6x^5 + \cdots$$
$$= (a_0 + a_1x + a_2x^2 + \cdots a_6x^6 + \cdots)(1 - x^2 + x^4 - x^6 + \cdots).$$

Equating coefficients of like powers of $x$,

$$a_1 = a_0, \quad 2a_2 = a_1, \quad 3a_3 = a_2 - a_0, \quad 4a_4 = a_3 - a_1, \quad 5a_5 = a_4 - a_2 + a_0, \quad 6a_6 = a_5 - a_3 + a_1.$$

Equation (1) by making $x = 0$ gives $a_0 = 1$ and the preceding equations give

$$a_1 = 1, \quad a_2 = \tfrac{1}{2}, \quad a_3 = -\tfrac{1}{6}, \quad a_4 = -\tfrac{7}{24}, \quad a_5 = \tfrac{1}{24}, \quad a_6 = \tfrac{29}{144}.$$

Hence,

$$e^{\tan^{-1}x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{7x^4}{24} + \frac{x^5}{24} + \frac{29x^6}{144} - \cdots.$$

$e^{\sin x}$ may be similarly developed.

$$e^{\sin x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_8x^8 + \cdots. \tag{2}$$

Differentiating,

$$e^{\sin x}\cos x = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + 8a_8x^7 + \cdots$$

$$= (a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_8x^8 + \cdots)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right).$$

Equating coefficients,

$$a_1 = a_0, \quad 2a_2 = a_1, \quad 3a_3 = a_2 - \frac{a_0}{2!}, \quad 4a_4 = a_3 - \frac{a_1}{2!}, \quad 5a_5 = a_4 - \frac{a_2}{2!} + \frac{a_0}{4!},$$

$$6a_6 = a_5 - \frac{a_3}{2!} + \frac{a_1}{4!}, \quad 7a_7 = a_6 - \frac{a_4}{2!} + \frac{a_2}{4!} - \frac{a_0}{6!}, \quad 8a_8 = a_7 - \frac{a_5}{2!} + \frac{a_3}{4!} - \frac{a_1}{6!}.$$

Making $x = 0$ in equation (2) gives $a_0 = 1$ and the preceding equations give

$$a_1 = 1, \quad a_2 = \tfrac{1}{2}, \quad a_3 = 0, \quad a_4 = -\tfrac{1}{8}, \quad a_5 = -\tfrac{1}{15}, \quad a_6 = -\tfrac{1}{240}, \quad a_7 = \tfrac{1}{90}, \quad a_8 = \tfrac{31}{5760}.$$

Hence,

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} - \frac{x^6}{240} + \frac{x^7}{90} + \frac{31x^8}{5760} + \cdots.$$

To develop $\tan x$, we notice that $\tan(-x) = -\tan x$. Hence, the expansion will contain only odd powers of $x$.

Assume

$$\tan x = a_1x + a_3x^3 + a_5x^5 + a_7x^7 + a_9x^9 + \cdots.$$

Differentiating,

$$\sec^2 x = a_1 + 3a_3x^2 + 5a_5x^4 + 7a_7x^6 + 9a_9x^8 + \cdots = 1 + \tan^2 x$$

$$= 1 + (a_1x + a_3x^3 + a_5x^5 + a_7x^7 + a_9x^9 + \cdots)^2.$$

Equating coefficients of like powers of $x$,

$$a_1 = 1, \quad 3a_3 = a_1^2, \quad 5a_5 = 2a_1a_3, \quad 7a_7 = 2a_1a_5 + a_3^2, \quad 9a_9 = 2a_1a_7 + 2a_3a_5.$$

From these, we obtain

$$a_1 = 1, \quad a_3 = \tfrac{1}{3}, \quad a_5 = \tfrac{2}{15}, \quad a_7 = \tfrac{17}{315}, \quad a_9 = \tfrac{62}{2835}.$$

Hence,

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots.$$

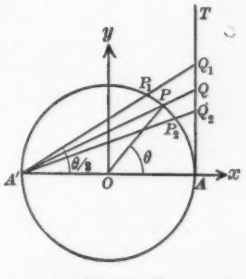
Also solved by E. D. GRANT, H. L. OLSON, J. L. RILEY, and the Proposer.

**2738 [January, 1919]. Proposed by W. D. CAIRNS, Oberlin College.**

Prove that between any two points on a unit circle with its center at the origin there is a point whose coördinates are rational.

SOLUTION BY P. J. DANIELL, Rice Institute.

Let $A'OA$ be the diameter on the $x$-axis, and let $P_1$, $P_2$ be the two given points. Through $A$ draw $AT$ perpendicular to $A'OA$. Let $A'P_1$, $A'P_2$ intersect $AT$ in $Q_1$, $Q_2$. By the theory of irrational numbers between the points $Q_1$, $Q_2$ on $AT$ there is a point $Q$ such that $AQ$ is rational and indeed equal to $2m/n$, where $m$, $n$ are integers. Let $A'Q$ intersect the circle in $P$. Then $P$ is the required point. It is assumed, and this involves no loss in generality, that $P_1$, $P_2$ lie on the same side of the $x$-axis. Then $P$ lies between $P_1$ and $P_2$.




$$x = \cos\theta = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} = \frac{1 - \frac{AQ^2}{A'A^2}}{1 + \frac{AQ^2}{A'A^2}} = \frac{1 - \frac{m^2}{n^2}}{1 + \frac{m^2}{n^2}} = \frac{n^2 - m^2}{n^2 + m^2}$$

is rational. Similarly $y = \dfrac{2mn}{n^2 + m^2}$ is rational.

Also solved by R. A. JOHNSON, H. L. OLSON, A. PELLETIER, W. R. RANSOM, J. ROSENBAUM, E. SWIFT, and the Proposer.

## NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

Mr. I. ROMAN, now studying at the University of Chicago, will return to Northwestern University as instructor in mathematics this fall.

Mr. P. W. HILL has returned from overseas' service and will again be instructor in mathematics at Wabash College.

Professor E. G. BILL has recently been advanced to the office of Director, Military Service Branch, by the Canadian Government. He expects to return to Dartmouth College this fall.

Dr. M. G. SMITH has been appointed professor of mathematics at Greenville College.

Mr. H. L. SMITH, of Cornell University, has been appointed instructor in mathematics and astronomy at Trinity College.

Dr. C. A. FISCHER, of Columbia University, has been appointed professor of mathematics and astronomy at Trinity College.

Dr. J. R. KLINE, instructor in mathematics during the past year has been appointed an "associate" in mathematics at the University of Illinois.

Professor E. B. VAN VLECK is to be a lecturer at Harvard University during the second semester of 1919–20.

Mr. H. R. PHALEN has been promoted to the position of assistant professor of mathematics at the Armour Institute of Technology.

Assistant Professor H. L. SLOBIN, of the University of Minnesota, has been appointed head of the department of mathematics in New Hampshire College.

At Lehigh University Mr. M. S. KNEBELMAN has been promoted to be assistant professor of mathematics.

Professor IDA BARNEY, of Lake Erie College, Painesville, O., has been appointed in charge of the department of mathematics at Meredith College, Raleigh, N. C.

Professor E. E. MOOTS, of Cornell College, Iowa, has been appointed head of the department of mathematics and engineering.

Professor H. M. SHOWMANN, of Colorado School of Mines, and lecturer in physics at Harvard during the past year, has been appointed assistant professor of mathematics at the Case School of Applied Science.

In accordance with a general policy for encouraging research, initiated in the latter part of June at the Massachusetts Institute of Technology, three members of the department of mathematics (not above the rank of assistant professor) are to be relieved of one-third of their teaching assignments next year and are expected to devote the time released to research. In order to make this possible Dr. S. D. ZELDIN, who has been instructor in the College of Hawaii and professor at Olivet College, has been appointed instructor in mathematics at the Institute.

*Science* announces that Dr. E. D. ROE, Jr., professor of mathematics at Syracuse University, has been elected director of the observatory. His position in the department of mathematics remains unchanged.

Professor ROBERT W. WILLSON has just become professor emeritus at Harvard University. He was assistant professor of astronomy 1899–1903, and has been professor of astronomy since 1903.

Professor N. E. NÖRLUND, of the University of Lund, has been appointed professor of mathematics at the University of Copenhagen.

The eightieth birthdays of the following mathematicians have occurred recently: LEO KOENIGSBERGER (November 15, 1917), THEODOR REYE (June 20, 1918), and HIERONYMUS ZEUTHEN (February 15, 1919). (*L'Enseignement mathématique* states that Zeuthen died on his birthday.) If MORITZ CANTOR was alive August 23, 1919 he would be ninety years old.

JOHN WILLIAM STRUTT, Lord RAYLEIGH, died June 30, 1919, aged 76.

Don EDUARDO TORROJA, professor of applied geometry at the University of Madrid died September 14, 1918 in the seventy-second year of his age.

PAUL MANSION, professor of mathematics at the University of Ghent, died April 18, 1919, aged 75. He was the author of scores of books and articles dealing with topics in higher algebra, theory of numbers, foundations of geometry and history of mathematics. For forty years he was co-editor of *Nouvelle Correspondance Mathématique* and its successor *Mathesis*.

Professor CHRISTOFORO ALASIA, whose biography and portrait appeared in this MONTHLY, August–September, 1902, died November 19, 1918, aged 49 years. He is probably known in this country chiefly as the author of *La recente geometria del triangolo* (Città di Castello, 1900) and of *566 Relazioni metriche e trigonometriche fra gli elementi d'un triangolo piano* (ibid., 1900), and as editor of *Le Matematiche* (2 vols., 1901–1903).

Professor O. D. KELLOGG has been made a fellow of the American Association for the Advancement of Science, and elected vice-president and chairman of Section A.

In the proposed organization of an International Astronomical Union the following members of the ASSOCIATION are members of committees in the organization of the American section: E. P. ADAMS, L. A. BAUER, P. P. BOYD, E. W. BROWN, W. J. HUSSEY, FRANK MORLEY, and F. R. MOULTON.

A Norwegian Mathematical Society was founded at Christiania on November 2, 1918 when about one hundred and fifty mathematicians came together from all parts of Norway. This society will meet in larger cities, and will publish a periodical, of which Professor HEEGAARD (recent editor of *Nyt Tidsskrift for Matematik*) and Oberlehrer ALEXANDER have been appointed editors.

From *Nature* we learn of a project for the creation and endowment of a geophysical institute at Cambridge, England, which is being promoted by a committee of prominent British scientists. It is felt that such an institute could render great assistance in connection with geodetic work which forms the basis of all the state surveys, and with the practical problems associated both with leveling, mean sea-level, and vertical movement of the earth's crust, and with a more thorough study of the tides and tide prediction than has ever been made.

Dr. VITO VOLTERRA, professor of mathematical physics in the University of Rome, will deliver a series of six lectures on the Hitchcock Foundation at the University of California in August or September.

At the Colloquium of the American Mathematical Society to be held at the University of Chicago in September, 1920, Professor G. D. BIRKHOFF is to lecture on "Dynamical Systems," and Professor F. R. MOULTON on "Certain topics in functions of infinitely many variables."

In conferring the degree of Doctor of Science on Professor STRATTON at the recent Yale Commencement the public orator spoke as follows: Samuel Wesley Stratton; Mathematician, physicist, professor in the Universities of Illinois and Chicago, a naval officer in the Spanish war, since 1891 Director of the National Bureau of Standards in weights and measures. Dr. Stratton's work in this bureau has been conspicuous and constructive, recognized beyond our own limits, vitally important in war and war research. A man weighed in the balance and not found wanting.

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## MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The Maryland-Virginia-District of Columbia Section of the Mathematical Association of America met at the U. S. Naval Academy, Annapolis, Maryland, May 3, 1919. Among those in attendance were the following members: R. N. Ashmun, International Boundary Commission; H. G. Avers, U. S. Coast and Geodetic Survey; Clara L. Bacon, Goucher College; C. C. Bramble, U. S. N. A.; J. A. Bullard, U. S. N. A.; Paul Capron, U. S. N. A.; G. R. Clements, U. S. N. A.; A. Cohen, Johns Hopkins University; G. H. Cresse, U. S. N. A.; L. S. Dederick, U. S. N. A.; Alexander Dillingham, U. S. N. A.; J. B. Eppes, U. S. N. A.; J. N. Galloway, U. S. N. A.; H. C. Gossard, U. S. N. A.; Angelo Hall, U. S. N. A.; W. M. Hamilton, U. S. Nautical Almanac Office; H. L. Hodgkins, George Washington University; L. S. Hulburt, Johns Hopkins University; W. W. Johnson, U. S. N. A.; A. E. Landry, Catholic University; J. J. Luck, University of Virginia; E. S. Mayer, U. S. N. A.; F. D. Murnaghan, Johns Hopkins University; J. R. Musselman, U. S. Food Commission; C. H. Rawlins, Jr., U. S. N. A.; M. R. Richardson, U. S. N. A.; R. E. Root, U. S. N. A.; W. F. Shenton, U. S. N. A.; Clara E. Smith, Goucher College; H. Ivah Thomsen, Johns Hopkins University; John Tyler, U. S. N. A.

The program consisted of a forenoon and an afternoon session, and contained the following papers:

### Forenoon Session—Symposium on "Mathematics for Engineering Students."

(1) Paper, Prof. Abraham Cohen, Johns Hopkins University (Mathematics);
(2) Discussion, Prof. Alexander Dillingham, U. S. N. A. (Mathematics);
(3) Paper, Prof. T. J. MacKavanagh, Catholic University (Engineering), by invitation;
(4) Discussion, Prof. L. A. Doggett, U. S. N. A. (Engineering), by invitation;
(5) General Discussion.

### Afternoon Session.

(6) "Circular parts, the general case."—Prof. W. W. Johnson, U. S. N. A.
(7) "A study of map projections in general."—Mr. Oscar S. Adams, U. S. Coast and Geodetic Survey.

The visitors in attendance were the guests of the members of the Naval Academy Department of Mathematics at a delightful luncheon served at one o'clock at The Sign of the Goat Inn.

At the beginning of the afternoon session the following officers were elected for the ensuing year:

Chairman, Professor R. E. Root, U. S. Naval Academy.